Problems and Remarks

1. Problem.

Consider functional equations of the form

$$\sum_{i=1}^{n} a_i f(\sum_{k=1}^{n_i} b_{ik} x_k) = 0, \qquad \sum_{i=1}^{n} a_i \neq 0 \tag{1}$$

and

$$\sum_{i=1}^{m} \alpha_i f(\sum_{k=1}^{m_i} \beta_{ik} x_k) = 0, \qquad \sum_{i=1}^{m} \alpha_i \neq 0, \tag{2}$$

where all parameters are real and $f\colon \mathbb{R} \to \mathbb{R}$.

Assume that the two functional equations are equivalent, i.e., they have the same set of solutions.

Can we say something about the common stability? More precisely, if (1) is stable, what can we say about the stability of (2). Under which additional conditions the stability of (1) implies that of (2)?

Gian Luigi Forti

2. Problem.

Let X be a Hilbert space, $D \subseteq X$ an open convex set, $\varepsilon > 0$ and let $f\colon D \to \mathbb{R}$ be a continuous function such that

$$f(tx + (1-t)y) - tf(x) - (1-t)f(y) \leq \varepsilon t(1-t)\|x - y\|, \qquad x, y \in D,\ t \in [0,1].$$

Does there exist an $x_0 \in D$ such that f is differentiable at x_0?

This problem is motivated by the results of S. Rolewicz.

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3. Problem and Remark.

Let X be a normed space, $D \subseteq X$ be an open convex set and let $f : D \to \mathbb{R}$ be a Lipschitz perturbation of a convex function $g : D \to \mathbb{R}$, i.e., let f be of the form

$$f = g + \ell,$$

where g is a convex function and $\ell : D \to \mathbb{R}$ is ε-Lipschitz, i.e.,

$$|\ell(x) - \ell(y)| \leq \varepsilon\|x - y\|, \qquad x, y \in D.$$

Then, for $x, y \in D$ and $t \in [0,1]$, we have

$$f(tx + (1-t)y) - tf(x) - (1-t)f(y) = [g(tx + (1-t)y) - tg(x) - (1-t)g(y)]$$
$$+[\ell(tx + (1-t)y) - t\ell(x) - (1-t)\ell(y)] \leq t[\ell(tx + (1-t)y) - \ell(x)]$$

$$+(1-t)[\ell(tx+(1-t)y)-\ell(y)] \leq t|\ell(tx+(1-t)y)-\ell(x)|+(1-t)|\ell(tx+(1-t)y)-\ell(y)|$$

$$\leq t\varepsilon\|(tx+(1-t)y)-x\|+(1-t)\varepsilon\|(tx+(1-t)y)-y\| = 2\varepsilon t(1-t)\|x-y\|.$$

Therefore, f satisfies the approximate convexity inequality:

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)+2\varepsilon t(1-t)\|x-y\|. \tag{1}$$

On the other hand, in the case $X = \mathbb{R}$, we have the following converse of the above observation (which is a particular case of a result obtained in [1]).

Proposition.
Let I be an open interval and $\varepsilon \geq 0$. Assume that $f : I \to \mathbb{R}$ satisfies, for all $x, y \in I$ and $t \in [0, 1]$, inequality (1). Then there exists a convex function $g : I \to \mathbb{R}$ such that the function $\ell := f - g$ is (2ε)-Lipschitz.

The following more general and open problem seems to be of interest.

Problem
Does there exist a constant γ (that may depend on X and D) such that, whenever a function $f : D \to X$ satisfies inequality (1) for all $x, y \in D$ and $t \in [0, 1]$, then there exists a convex function $g : D \to \mathbb{R}$ such that the function $\ell := f - g$ is $\gamma\varepsilon$-Lipschitz on D?

A result related to this problem was stated by V. Protasov during the 13th ICFEI:

If a function $f : X \to \mathbb{R}$ satisfies, for all $x, y \in X$ and $t \in [0, 1]$,

$$|f(tx+(1-t)y)-tf(x)-(1-t)f(y)| \leq 2\varepsilon t(1-t)\|x-y\|,$$

then there exists a continuous linear functional $x^* \in X^*$ such that $\ell := f - x^*$ is (4ε)-Lipschitz on X.

Reference

[1] Zs. Páles, *On approximately convex functions*, Proc. Amer. Math. Soc. **131** (2003), 243–252.

Zsolt Páles

4. Problem.
In connection with some problem in theoretical physics, O.G. Bokov introduced in [1] the following functional equation

$$f(x,y)f(x+y,z)+f(y,z)f(y+z,x)+f(z,x)f(z+x,y) = 0. \tag{1}$$

In [2] A.V. Yagzhev determined all analytic solutions $f : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ of (1). However, his proof is not clear and presents several gaps. So, we may wonder about the validity of the result. Therefore, the problem is to find all analytic solutions $f : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ of (1) with a nice mahematical proof. Also, we may ask about the solutions of (1) in a more general setting.

References

[1] O.G. Bokov, *A model of Lie fields and multiple-time retarded Green's functions of an electromagnetic field in dielectric media*, Nauchn. Tr. Novosib. Gos. Pedagog. Inst. **86** (1973), 3–9.

[2] A.V. Yagzhev, *A functional equation from theoretical physics*, Funct. Anal. Appl. **16** (1982), 38–44.

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5. Remark.

During the last fifteen years a great number of papers concerning stability of functional equations have been published. Unfortunately in many of these papers motivations for studying a given equation or/and possible applications of the stability results are missing. In my opinion this will eventually produce a discredit of the topic and, consequently, a discredit of the field of functional equations: a thing that we, functional equationists, certainly do not want. These considerations are mainly directed to younger colleagues, in order to invite them to investigate genuine, not rather artificial, mathematical problems.

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6. Remark.

Let $(X, \|\cdot\|)$ be a normed space, $D \subset X$ be a convex set and $c > 0$ be a fixed constant. A function $f : D \to \mathbb{R}$ is called *strongly convex with modulus c* if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) - ct(1-t)\|x-y\|^2 \tag{1}$$

for all $x, y \in D$ and $t \in [0,1]$. Under the assumption (A) that $(X, \|\cdot\|)$ is an inner product space, the following equivalence (B) holds:

$f : D \to \mathbb{R}$ *is strongly convex with modulus c if and only if* $g = f - c\|\cdot\|^2$ *is convex.*

The following example gives an answer to the question posed by Zsolt Páles after my talk at this conference and shows that assumption (A) is essential for (B).

Example.
Let $X = \mathbb{R}^2$ and $\|x\| = | x_1 | + | x_2 |$ for $x = (x_1, x_2)$. Take $f = \|\cdot\|^2$. Then $g = f - \|\cdot\|^2 = 0$ is convex. However, f is not strongly convex with modulus 1. Indeed, for $x = (1,0)$ and $y = (0,1)$ we have

$$f\Big(\frac{x+y}{2}\Big) = 1 > 0 = \frac{f(x)+f(y)}{2} - \frac{1}{4}\|x-y\|^2,$$

which contradicts (1).

One can also prove that if (B) holds for every $f : X \to \mathbb{R}$, then $(X, \|\cdot\|)$ must be an inner product space. Thus condition (B) gives another characterization of the inner product spaces among normed spaces.

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