Abstracts of Talks

Roman Badora
Silesian University, Katowice, Poland

Stability of some functional equations

Let X be a group and let Λ be a finite subgroup of the automorphism group of X ($N = \mathrm{card}\Lambda$ and the action of $\lambda \in \Lambda$ on $x \in X$ is denoted by λx). We study the stability of the following functional equations

$$\frac{1}{N}\sum_{\lambda\in\Lambda} f(x+\lambda y) = f(x)g(y) + h(y), \quad x, y \in X;$$

$$\frac{1}{N}\sum_{\lambda\in\Lambda} f(x+\lambda y) = f(y)g(x) + h(x), \quad x, y \in X$$

($f, g, h : X \to \mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$), which cover Jensen's functional equation, Cauchy's functional equation, the exponential functional equation, the functional equation of the square of the norm and d'Alembert's functional equation.

Anna Bahyrycz
Pedagogical University, Kraków, Poland

On systems of equations with unknown multifunctions

Let $(G, +)$ be a grupoid, T be a nonempty set. Inspired by problem posed by Z. Moszner in [1] we investigate for which additional assumptions putting on the multifunctions $Z(t) : T \to 2^G$ which satisfy condition

$$\bigcup_{t\in T} Z(t) = G$$

and system of conditions

$$(\exists_{t\in T}\, i(t)j(t) \neq 0) \Rightarrow (\bigcap_{t\in T} Z(t)^{i(t)} + \bigcap_{t\in T} Z(t)^{j(t)} \subset \bigcap_{t\in T} Z(t)^{i(t)j(t)}), \tag{1}$$

where $Z(t)^1 := Z(t), Z(t)^0 := G \setminus Z(t)$ and $i(t), j(t) : T \to \{0, 1\}$ are the arbitrary functions not identically equal to zero, the inclusion in the above conditions (1) may be replaced by equality, obtaining the system of equations with unknown multifunctions.

Reference

[1] Z. Moszner, *Sur la fonction de choix et la fonction d'indice*, Ann. Acad. Pedagog. Crac. Stud. Math. **4** (2004), 143–169.

Szabolcs Baják
University of Debrecen, Hungary
(joint work with **Zs. Páles**)

Invariance equations for Gini and Stolarsky means

Given three strict means $M, N, K : \mathbb{R}_+^2 \to \mathbb{R}_+$, we say that the triple (M, N, K) satisfies the *invariance equation* if

$$K(M(x, y), N(x, y)) = K(x, y), \qquad x, y \in \mathbb{R}_+$$

holds. It is well known that K is uniquely determined by M and N, and it is called the Gauss composition $K = M \otimes N$ of M and N.

Our aim is to solve the invariance equation when each of the means M, N, K is either a Gini or a Solarsky mean with different parameters, thus we have to consider four different equations. With the help of the computer algebra system Maple V Release 9, we give the general solutions of these equations.

Karol Baron
Silesian University, Katowice, Poland

On Baire measurable solutions of some functional equations

We establish conditions under which Baire measurable solutions f of

$$\Gamma(x, y, |f(x) - f(y)|) = \Phi(x, y, f(x + \varphi_1(y)), \ldots, f(x + \varphi_N(y)))$$

defined on a metrizable topological group are continuous at zero.

Svetlana S. Belmesova
Nizhny Novgorod State University, Russia
(joint work with **L.S. Efremova**)

On the unbounded invariant curves of some polynomial maps

The unbounded trajectories of the quadratic mapping $F_2(x,\, y) = (xy,\, (x-2)^2)$ in the plane $\mathbb{R}^2$ has been studied in [1].

In this work we deal with the one-parameter family of the quadratic mappings

$$F_\mu(x, y) = (xy, (x - \mu)^2), \tag{1}$$

where $(x, y) \in \mathbb{R}^2$, $\mu \in (0, 1]$. It is proved the existence of the unbounded invariant curves for the mappings (1) for every $\mu \in (0, 1]$.

Reference

[1] S.S. Belmesova, L.S. Efremova, *On unbounded trajectories of a certain quadratic mapping of the plane*, J. Math. Sci. (N. Y.) **157** (2009), 433–441.

Mihály Bessenyei
University of Debrecen, Hungary

On a class of single variable functional equations

In the last few years, functional equations have had a growing importance in competitions for secondary school students in Hungary (browse the issues of Mathematical and Physical Journal for Secondary Schools). A typical exercise is of the form

$$\alpha_1 f \circ g_1 + \cdots + \alpha_n f \circ g_n = h,$$

where g_k, α_k, h, f are given functions (with appropriate domain and range) under the assumption that $g_1, \ldots, g_n$ generate a group under the operation of composition. The main results of the present talk guarantee that, under some reasonable assumptions, the functional equation above (and also its nonlinear correspondence) has a unique solution. The proofs are based on Cramer's rule and the inverse-function theorem.

References

[1] *Mathematical and Physical Journal for Secondary Schools (KöMal)* (http://www.komal.hu).

[2] V.S. Brodskii, A.K. Slipenko, *Functional equations*, Visa Skola, Kiev, 1986 (in Russian).

[3] K. Lajkó, *Functional equations in exercises*, University Press of Debrecen, 2005 (in Hungarian).

Zoltán Boros
University of Debrecen, Hungary

Inequalities for pairs of additive functions

Representation theorems are presented for pairs of additive functions, under the assumption that a related expression is locally bounded. Let us assume that f and g are real additive functions. If

$$\frac{1}{x} f(x) + x g\left(\frac{1}{x}\right)$$

is bounded on a non-void open interval or

$$x f(x) + \sqrt{1-x^2}\, g(\sqrt{1-x^2})$$

is bounded on every compact subinterval of the open interval $(0, 1)$, then there exists a real derivation d such that

$$f(x) = d(x) + f(1)x \quad \text{and} \quad g(x) = d(x) + g(1)x$$

for every real number x. However, if, for instance,

$$\sqrt{1-x^2}f(x) - xg(\sqrt{1-x^2})$$

is bounded on every compact subinterval of the open interval $(0,1)$, then f and g are linear.

Nicole Brillouët-Belluot

Ecole Centrale de Nantes, France

(joint work with **J. Chudziak** and **J. Brzdęk**)

Some further results concerning a conditional Gołąb-Schinzel equation

Let X be a real linear space and let $M : \mathbb{R} \to \mathbb{R}$ be a continuous and multiplicative function. We determine the solutions $f : X \to \mathbb{R}$ of the functional equation

$$f(x + M(f(x))y)f(x)f(y)[f(x + M(f(x))y) - f(x)f(y)] = 0$$

which are continuous on rays, i.e. which are such that, for every $x \in X \setminus \{0\}$, $f_x : \mathbb{R} \to \mathbb{R}$ defined by $f_x(t) = f(tx)$ is continuous.

In the particular cases where $M \equiv 1$ and $M(x) \equiv x$, we obtain the continuous on rays solutions of a conditional exponential equation and those of a conditional Gołąb-Schinzel equation.

These results extend those given by the authors at the 47th ISFE in Gargnano.

Janusz Brzdęk

Pedagogical University, Kraków, Poland

(joint work with **D. Popa** and **B. Xu**)

On nonstability of the linear recurrence of order one

Let $\mathbb{K}$ be either the field of reals or the field of complex numbers, X be a Banach space over $\mathbb{K}$, $(a_n)_{n\geq 0}$ a sequence in $\mathbb{K} \setminus \{0\}$, and $(b_n)_{n\geq 0}$ a sequence in X. We present a result concerning nonstability of the linear recurrence

$$y_{n+1} = a_n y_n + b_n, \quad n \geq 0.$$

This corresponds to the contents, e.g., of recent papers [1–5].

References

[1] J. Brzdęk, D. Popa, B. Xu, *Note on nonstability of the linear recurrence*, Abh. Math. Sem. Univ. Hamburg **76** (2006), 183–189.

[2] J. Brzdęk, D. Popa, B. Xu, *The Hyers-Ulam stability of nonlinear recurrences*, J. Math. Anal. Appl. **335** (2007), 443–449.

[3] J. Brzdęk, D. Popa, B. Xu, *Hyers-Ulam stability for linear equations of higher orders*, Acta Math. Hungar. **120** (2008), 1–8.

[4] D. Popa, *Hyers-Ulam-Rassias stability of a linear recurrence*, J. Math. Anal. Appl. **309** (2005), 591–597.

[5] T. Trif, *On the stability of a general gamma-type functional equation*, Publ. Math. Debrecen **60** (2002), 47–61.

Pál Burai
University of Debrecen, Hungary
(joint work with **A. Házy**)

Some results on Orlicz-convex functions

Let X be a linear space over the real field $\mathbb{R}$, and $\mathcal{C} \subset X$ be an open, nonempty cone. A function $f : \mathcal{C} \to \mathbb{R}$ is called s-convex (Orlicz-convex) if

$$f\left(\lambda^s x + (1-\lambda)^s y\right) \leq \lambda f(x) + (1-\lambda) f(y)$$

for all $x, y \in \mathcal{C}\,,\ \lambda \in (0,1]$, where $s \in [1,\infty)$ is fixed number. In this talk we make some examination in this class of functions.

Liviu Cădariu
"Politehnica" University of Timişoara, Romania

Remarks on the fixed point method for Ulam-Hyers stability

In [1] and [2] some generalized Ulam-Hyers stability results for Cauchy functional equation have been proved. One of the results reads as follows:

Let us consider a real linear space E, a complete p-normed space F and a sub-homogenous functional of order α $||(\cdot,\cdot)||_\alpha : E \times E \to [0,\infty)$, with $\alpha \neq p$. In these conditions, the following stability property holds: For each $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for every mapping $f : E \to F$ which satisfies

$$||f(x) + f(y) - f(x+y)||_p \leq \delta(\varepsilon) \cdot ||(x,y)||_\alpha, \qquad x, y \in E,$$

there exists a unique additive mapping $a : E \to F$ such that

$$||f(x) - a(x)||_p \leq \varepsilon \cdot ||(x,x)||_\alpha, \qquad x \in E.$$

We intend to outline the results concerning the generalized Ulam-Hyers stability for different other kinds of functional equations. Both the Hyers direct method and the fixed point method will be emphasized and we shall consider functions defined on linear spaces and taking values in p-normed spaces or random normed spaces.

References

[1] L. Cădariu, *A general theorem of stability for the Cauchy's equation*, Bull. Ştiinţ. Univ. Politeh. Timiş. Ser. Mat. Fiz. **47(61)** (2002), 14–28.

[2] L. Cădariu, V. Radu, *On the stability of the Cauchy functional equation: a fixed points approach*, Iteration theory (ECIT'02), 43–52, Grazer Math. Ber. **346**, Karl-Franzens-Univ. Graz, Graz, 2004.

[3] D.H. Hyers, G. Isac, Th.M. Rassias, *Stability of functional equations in several variables*, Progress in Nonlinear Differential Equations and their Applications **34**, Birkhäuser Boston, Inc., Boston, MA, 1998.

[4] V. Radu, *The fixed point alternative and the stability of functional equations*, Fixed Point Theory **4** (2003), 91–96.

Jacek Chmieliński
Pedagogical University, Kraków, Poland

Stability of linear isometries and orthogonality preserving mappings

In reference to a question posed by the author during the 12th ICFEI, a short survey on *linear approximate isometries* in normed spaces and respective stability problems will be given.

Next, an application to the problem of stability of orthogonality preserving mappings in normed spaces will be shown. Results from a joint work with **P. Wójcik** will be presented.

Jacek Chudziak
University of Rzeszów, Poland

Stability of a composite functional equation

At the 47th International Symposium on Functional Equations (Gargnano, Italy) J. Brzdęk has posed several questions concerning a quotient stability of the following generalization of the Gołąb-Schinzel functional equation

$$f(x + M(f(x))y) = f(x)f(y).$$

In our talk we present the answers for some of them.

Krzysztof Ciepliński
Pedagogical University, Kraków, Poland

Stability of the multi-Jensen equation

Assume that V is a normed space, W is a Banach space and $m \geq 2$ is an integer. A function $f : V^m \to W$ is called *multi-Jensen* (we also say that f satisfies multi-Jensen equation) if it is a Jensen mapping in each variable, that is

$$f(x_1, \ldots, x_{i-1}, \frac{1}{2}(x_i + y_i), x_{i+1}, \ldots, x_m) =$$

$$\frac{1}{2}f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_m) + \frac{1}{2}f(x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_m),$$

$$i \in \{1, \ldots, m\},\ x_1, \ldots, x_i, y_i, \ldots, x_m \in V.$$

This notion was introduced by W. Prager and J. Schwaiger in 2005 with the connection with generalized polynomials (see [1]).

In this talk the stability of multi-Jensen equation is discussed.

Reference

[1] W. Prager, J. Schwaiger, *Multi-affine and multi-Jensen functions and their connection with generalized polynomials*, Aequationes Math. **69** (2005), 41–57.

Stefan Czerwik

Silesian University of Technology, Gliwice, Poland

S.M. Ulam – his life and results in mathematics, physics and biology

We shall present the information about the life of S. M. Ulam and his results in different areas of science: mathematics, physics and biology; particularly in stability of functional equations and H-bomb.

Zoltán Daróczy

University of Debrecen, Hungary

(joint work with **Zs. Páles**)

On an elementary inequality and conjugate means

Let $n \geq 2$, $k \geq 1$. In this talk we give the necessary and sufficient condition for the real numbers $p_1, p_2, ..., p_n, q_1, q_2, ..., q_k$ to fulfill the following property:

If

$$\min\{x_i\} \leq M_l \leq \max\{x_i\}, \qquad l = 1, 2, ..., k$$

holds for all real numbers $x_1, x_2, ..., x_n$ and $M_1, M_2, ..., M_k$, then

$$\min\{x_i\} \leq \sum_{i=1}^{n} p_i x_i + \sum_{l=1}^{k} q_l M_l \leq \max\{x_i\}.$$

Let I be a nonvoid open interval and let $M_l : I^n \to I$ $(l = 1, 2, ..., k)$ be means. If there exist $p_1, p_2, ..., p_n, q_1, q_2, ..., q_k$ with the property above and a strictly monotone, continuous function φ on I then

$$M(x_1, x_2, ..., x_n) = \varphi^{-1}\left(\sum_{i=1}^{n} p_i \varphi(x_i) + \sum_{l=1}^{k} q_l \varphi(M_l(x_1, x_2, ..., x_n))\right), \qquad x_1, x_2, ..., x_n \in I$$

is a mean value and we call it the *conjugate mean generated by the means* $M_1, M_2, ..., M_k$.

We deal with several problems on conjugate means.

Judita Dascăl
University of Debrecen, Hungary
(joint work with **Z. Daróczy**)

On conjugate means

Let $I \subset \mathbb{R}$ be a nonvoid open interval.

A function $M : I^2 \to I$ is said to be a *conjugate mean* on I if there exist real numbers $p, q \in [0, 1]$ and a continuous, strictly monotone real valued function φ defined on I such that

$$M(x, y) = \varphi^{-1}\left(p\varphi(x) + q\varphi(y) + (1 - p - q)\varphi\left(\frac{x + y}{2}\right)\right), \qquad x, y \in I.$$

We deal with the equality problem in the class of conjugate means.

Joachim Domsta
Gdańsk University of Technology, Poland

A comparison of quantum dynamical semigroups obtainable by mixing or partial tracing

Some simple examples of quantum systems are collected to illustrate requirements sufficient for the evolution of a subsystem according to a quantum dynamical semigroup. For this, a class of quantum dynamics of a system S coupled to a reservoir R is analyzed in the Hilbert space $\mathcal{H}_{SR} = \mathcal{H}_S \otimes \mathcal{H}_R$, where $\mathcal{H}_R = L^2(\mathbb{R})$ and $\mathcal{H}_S = l^2_I$, with I standing for a complete at most countable set of pure orthogonal states of S. The Hamiltonian of SR is built of tensor products of multipliers acting on $\mathcal{H}_S$ and $\mathcal{H}_R$. The chosen linear coupling implies the exponential decoherence of the reduced evolution of S if and only if the occupation density in R is of the Cauchy type. Then the system indicates the exponential decoherence. On the other hand, since the occupation density in S is discrete, the reduced evolution of R is never governed by a semigroup (unless there is no coupling).

In the considered case, the reduced evolution of the subsystem S as well as of the reservoir R can be equivalently obtained by taking the expectation (i.e. by averaging) of the unitary dynamics of the alone standing system S or R with suitably chosen random Hamiltonians. Thus again, the probability distribution of the random perturbation for S must be of the Cauchy type if the exponential decoherence should follow.

In the models of the third class the phase of the quantum system S varies according to a stochastic process with independent stationary increments. In other words, this is an example of a random dynamical system. Then the exponential decoherence of the evolution of the averaged state follows, independently of the distribution of the process. In such cases the Itô-Schrödinger equation for the random unitary dynamics and the master equation for the averaged density matrices are obtained in the dependence on the probability distribution of the process. For presenting the Cauchy distribution in a different context, a relation to the exponential decay of the autocorrelation of autonomous systems is discussed briefly.

Andrey S. Filchenkov
Nizhny Novgorod State University, Russia
(joint work with **L.S. Efremova**)

On the simplest topologically transitive skew products in the plane

Let $F(x, y) = (f(x), g_x(y)) : I \to I$ be a skew product of interval maps, I is a rectangle in the plane, $I = I_1 \times I_2$ (I_1, I_2 are closed intervals). Let $T^1(I)$ be the space of $C^1(I)$-smooth skew products of interval maps.

In this talk we present conditions under which the set of periodic points of the skew product is dense in the phase space.

Theorem.
Let $F \in T^1(I)$ satisfy the following conditions:
1) $F(x, y)$ is a topologically transitive skew product of interval maps;
2) the partial derivative $\frac{\partial g_x(y)}{\partial y}$ monotonically decreases with respect to $y \in I_2$ for any $x \in I_1$;
3) $g_x(\partial I_2) = \partial I_2$ for any $x \in I_1$, where ∂I_2 is the boundary of I_2.
Then the set of periodic points of the skew product of interval maps is dense in I.

We also construct the topologically transitive skew product which satisfies all conditions of the above theorem. We use here the unimodal maps theory (see [2]). In [3] it is proved the existence of the topologically transitive cylindrical cascade (the skew product over the irrational rotation of the circle) without periodic points. On the other hand, in [1] it is constructed an example of continuous but not smooth topologically transitive skew product in the unit square which has the dense set of periodic points in horizontal fibers $y = 0$ and $y = 1$.

References

[1] Ll. Alseda, S. Kolyada, J. Llibre, L. Snoha, *Entropy and periodic points for transitive maps*, Trans. Amer. Math. Soc. **351** (1999), 1551–1573.

[2] L.S. Efremova, A.S. Filchenkov, *About one example of the topologically transitive skew product of interval maps in the plane*, Math problems, M.:MPhTI 2009, 61–68.

[3] E.A. Sidorov, *Topologically transitive cylindrical cascades* (Russian), Mat. Zametki **14** (1973), 441–452.

Gian Luigi Forti
Università degli Studi di Milano, Italy

Symbolic dynamics generated by graphs

In many natural phenomena strings consisting of sequences of symbols play a central role. Also the evolution of large classes of dynamical systems can be described, under certain conditions, as a sequence of symbols. In this context, a central question is how to enumerate and to characterize the full set of possible sequences generated by a dynamical system.

At first, the properties of the symbolic dynamics generated by a graph on an alphabet are presented and it is shown that the number of sequences of length n is either exponential or polynomial with respect to n.

Then by a combination of several graphs we obtain different laws. In particular we can obtain laws observed in complex systems and conjectured in 1992 by Ebeling and Nicolis.

We finish by presenting a probabilistic approach to the problem.

Roman Ger

Silesian University, Katowice, Poland

On a problem of Cuculière

In the February 2008 issue of *The American Mathematical Monthly* (Problems and Solutions, p.166) the following question was proposed by R. Cuculière:

Find all nondecreasing functions f from $\mathbb{R}$ to $\mathbb{R}$ such that
$f(x + f(y)) = f(f(x)) + f(y)$ for all real x and y

(Problem **11345**).
We shall present:
– the general Lebesgue measurable solution,
– monotonic solutions,
– a description of the general solution
of the functional equation in question.

Attila Gilányi

University of Debrecen, Hungary

Conditional stability of monomial functional equations

During the 42nd International Symposium on Functional Equations in Opava, Czech Republic, 2004, J. Aczél announced the program of the investigation of conditional functional equations (c.f. [1]). Connected to this program, we present some conditional stability results for monomial functional equations.

More precisely, in the case of various sets $D \subseteq \mathbb{R} \times \mathbb{R}$ and $H \subseteq \mathbb{R}$, and assuming that Y is a Banach space, n is a positive integer, α is an arbitrary, ε and δ are nonnegative real numbers, we examine whether the validity of the inequality

$$\|\Delta_y^n f(x) - n!f(y)\| \leq \varepsilon|x|^\alpha + \delta|y|^\alpha, \qquad (x, y) \in D$$

implies the existence of nonnegative constants c and d and a monomial function $g : \mathbb{R} \to Y$ of degree n (i.e. a solution of the functional equation $\Delta_y^n g(x) - n!g(y) = 0,\ x, y \in \mathbb{R}$) for which

$$\|f(x) - g(x)\| \leq (c\varepsilon + d\delta)|x|^\alpha, \qquad x \in H$$

holds.

Reference

[1] J. Aczél, *5. Remark, Report of Meeting*, Aequationes Math. **69** (2005), 183.

Dorota Głazowska
University of Zielona Góra, Poland
(joint work with **J. Matkowski**)

An invariance of the geometric mean with respect to the Cauchy mean-type mappings

We consider the problem of invariance of the geometric mean with respect to the Cauchy mean-type mappings $\left(D^{f,g}, D^{h,k}\right)$, i.e., the functional equation

$$G \circ \left(D^{f,g}, D^{h,k}\right) = G.$$

Assuming that the generators g and k are power functions we show that the functions f and h have to be of high class of regularity. This fact allows to reduce the problem to differential equations and find some necessary conditions for generators f and h.

Eszter Gselmann
University of Debrecen, Hungary

On the stability of derivations

In this talk we investigate the stability of a system of functional equations that defines real derivations. More precisely, the problem of Ulam is considered in connection with the following system of equations

$$f(x+y) = f(x) + f(y), \qquad x \in \mathbb{R}$$

and

$$f(x^n) = cx^k f(x^m), \qquad x \in \mathbb{R} \setminus \{0\},$$

where $f : \mathbb{R} \to \mathbb{R}$ is the unknown function, $c \in \mathbb{R}$ and $n, m, k \in \mathbb{R}$ are arbitrarily fixed. Using a preliminary lemma that is also presented, it is proved that the above system of functional equations is stable in the sense of Hyers and Ulam, under some conditions on the parameters c, n, m and k.

Grzegorz Guzik
AGH University of Science and Techology, Kraków, Poland

On some disjoint iteration semigroups on the torus

General construction of measurable (continuous) disjoint iteration semigroups of triangular mappings on the torus is given.

Attila Házy
University of Miskolc, Hungary

Bernstein-Doetsch type results for h-convex functions

The concept of h-convexity was introduced by S. Varosanec in [1]. In our talk we introduce a more general concept of the h-convexity, and the concept of the so called (H, h)-convexity.

A h-convex (or (H, h)-convex) function is defined as a function $f : D \to \mathbb{R}$ (where D is a nonempty, open, convex subset of a real (or complex) linear space) which satisfies

$$f(\lambda x + (1 - \lambda)y) \le h(\lambda)f(x) + h(1 - \lambda)f(y),$$

for all $x, y \in D$ and $\lambda \in [0, 1]$ (resp. $\lambda \in H$), where h is a given real function.

The main goal of our talk is to prove some regularity and Bernstein-Doetsch type result for h-convex and (H, h)-convex functions. We also collect some facts on such functions. Finally, we collect some interesting, easily-proved properties of h-convex functions.

Reference

[1] S. Varošanec, *On h-convexity*, J. Math. Anal. Appl. **326** (2007), 303–311.

Eliza Jabłońska
Rzeszów University of Technology, Poland

About solutions of a generalized Gołąb–Schinzel equation

Let $n \in \mathbb{N}$ and let X be a metrizable linear space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We consider solutions $f : X \to \mathbb{K}$ of the functional equation

$$f(x + f(x)^n y) = f(x)f(y) \quad \text{for } x, y \in X$$

such that either f is bounded on a set of second category with the Baire property or f is Baire measurable. Our result generalizes a result of J. Brzdęk.

Hans-Heinrich Kairies
Clausthal University of Technology, Clausthal-Zellerfeld, Germany

A sum type operator

Our sum type operator $F : D \to F[D]$ is given by

$$F[\varphi](x) := \sum_{k=0}^{\infty} 2^{-k}\varphi(2^k x),$$

where $D = \{\varphi : \mathbb{R} \to \mathbb{R} \ ; \ \sum_{k=0}^{\infty} 2^{-k}\varphi(2^k x) \text{ converges for every } x \in \mathbb{R}\}$.

We treat the following aspects:

1. Historical background.
2. Basic properties of F and its restrictions $F_{rg} : D_{rg} \to F[D_{rg}]$ to sixteen subspaces D_{rg} of D, which are all vector spaces and in part Banach spaces.
3. Functional equations for $F[\varphi]$ and characterizations.
4. Some Fourier analysis for $F[\varphi]$.
5. Images $F[S]$ and $F^{-1}[S]$.
6. Eigenvalues and eigenspaces for all the sixteen F_{rg}.
7. Continuous and residual spectra.
8. Extensions.

Barbara Koclęga-Kulpa
Silesian University, Katowice, Poland
(joint work with **T. Szostok**)

On a class of equations stemming from various quadrature rules

We deal with a functional equation of the form

$$F(y) - F(x) = (y - x)\sum_{k=1}^{n} a_k f(\lambda_k x + (1-\lambda_k)y), \ \ x, y \in \mathbb{R} \tag{1}$$

motivated by quadrature rules of approximate integration. In previous results the solutions of this equation were found only in some particular cases. For example, coefficients λ_k were supposed to be rational or the equation in question was solved only for $n = 2$.

We prove that every function $f : \mathbb{R} \to \mathbb{R}$ satysfying equation (1) with some function $F : \mathbb{R} \to \mathbb{R}$, where $\sum_{k=1}^{n} a_k \neq 0$, is a polynomial of degree at most $2n - 1$. In our results we do not assume any specific form of coefficients occuring at the right-hand side of (1) and we allow n to be any positive integer. Moreover, we obtain solutions of our equation without any regularity assumptions concerning functions f and F.

Zygfryd Kominek
Silesian University, Katowice, Poland
(joint work with **J. Sikorska**)

On a Jensen-Hosszú equation

It is known that in the class of functions acting the interval $I = [0,1]$ ($I = (0,1)$) into a real Banach space the Jensen functional equation is stable and the Hosszú functional eqution has not this property. So, we have a nontrivial pair of the equivalent equations such that one of them is stable and the other is not. From this point of view it seems interesting to consider the functional equation of the form

$$f(x+y-xy)+f(xy)=2f(\frac{x+y}{2}), \qquad x,y\in I. \tag{1}$$

The left-hand-side of equation (1) is the same as the left-hand-side of the Hosszú functional equation, and the right-hand-side of our equation coincides with the left-hand side of the Jensen equation. We will prove that equation (1) is also equivalent to the Jensen (and in the same reason to the Hosszú) equation and, moreover, that equation (1) is stable in the sense of Hyers and Ulam.

Dorota Krassowska
University of Zielona Góra, Poland

On iteration semigroups containing generalized convex and concave functions

Let $I \subset \mathbb{R}$ be an open interval and let $M, N : I^2 \to I$ be continuous functions. A function $f : I \to I$ is said to be (M,N)-*convex* ((M,N)-*concave*) if

$$f(M(x,y)) \leq (\geq) N(f(x), f(y)), \qquad x,y \in I.$$

A function $f : I \to I$ simulteneously (M,N)-convex and (M,N)-concave is called (M,N)-*affine* (see [1]).

We prove that if in a continuous iteration semigroup $\{f^t : t \geq 0\}$ every element f^t is (M,N)-convex or (M,N)-concave and there exist $r > s > 0$ such that f^r and f^s are (M,N)-affine, then $M = N$ and every element of a semigroup is (M,M)-affine. We also consider the case where $M = N$ and we show that if in a continuous iteration semigroup $\{f^t : t \geq 0\}$ there exist $f^r <$ id and $f^s <$ id such that $\frac{r}{s} \notin \mathbb{Q}$ and f^r is (M,M)-convex and f^s is (M,M)-concave, then every element of the semigroup is (M,M)-affine.

Reference

[1] J. Matkowski, *Iteration groups with generalized convex and concave elements*, Iteration theory (ECIT 94) (Opava), 199–216, Grazer Math. Ber. **334**, Karl-Franzens-Univ. Graz, Graz, 1997.

Zbigniew Leśniak
Pedagogical University, Kraków, Poland

On conjugacy of Brouwer homeomorphisms

We consider Brouwer homeomorphisms of the plane for which the oscillating set is empty. The main result says that if the sets of indices of coverings of the plane consisting of maximal parallelizable regions for two Brouwer homeomorphisms are isomorphic and if for each of these regions there exists a one-to-one correspondence between the set of singular lines contained in the boundary of the region and the set of singular lines contained in the interior of the region, then these Brouwer homeomorphisms are conjugated. This theorem holds for Brouwer homeomorphisms that are embeddable in a flow as well as for Brouwer homeomorphisms for which there exists a foliation of the plane consisting of invariant topological lines.

Andrzej Mach
Jan Kochanowski University, Kielce, Poland
(joint work with **Z. Moszner**)

Stability of some functional equations and open problems

Some results on stability of certain equations and systems of equations are given.
A number of open problems of stability, raised by Z. Moszner, is presented. The answer for one of them is given.

References

[1] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A **27** (1941), 222–224.

[2] A. Mach, *On some functional equations involving Babbage equation*, Results Math. **51** (2007), 97–106.

[3] A. Mach, Z. Moszner, *On stability of the translation equation in some classes of functions*. Aequationes Math. **72** (2006), 191–197.

[4] A. Mach, Z. Moszner, *On some functional equations involving involutions*, Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II **216** (2007), 3–13 (2008).

[5] Z. Moszner, *On the stability of functional equations*, Aequationes Math. **77** (2009), 33–88.

Ewelina Mainka
Silesian University of Technology, Gliwice, Poland

On uniformly continuous Nemytskii operators generated by set-valued functions

Let $I = [0, 1]$, let Y be a real normed linear space, C a convex cone in Y and Z a Banach space. Denote by $\mathrm{clb}(Z)$ the set of all nonempty closed and bounded subsets of Z.

If a superposition operator N generated by a set-valued function $F\colon I \times C \to \mathrm{clb}(Z)$ maps the set $H_\alpha(I, C)$ of all functions $\varphi\colon I \to C$ satisfying the Hölder condition into the set

$H_\beta(I, \mathrm{clb}(Z))$ of all set-valued functions $\phi: I \to \mathrm{clb}(Z)$ satisfying the Hölder condition and is uniformly continuous, then

$$F(x,y) = A(x,y) \overset{*}{+} B(x), \quad x \in I,\ y \in C$$

for some set-valued functions A, B such that $A(\cdot, y), B \in H_\beta(I, \mathrm{clb}(Z))$, $y \in C$ and $A(x, \cdot) \in \mathcal{L}(C, \mathrm{clb}(Z))$, $x \in I$.

Using Jensen functional equation is essential in the proof. A converse result is also considered.

Judit Makó

University of Debrecen, Hungary
(joint work with **Zs. Páles**)

On φ-convexity

In this talk a new concept of approximate convexity is definied, termed *φ-convexity.* The function φ is chosen in a particular way. Assume that I is a nonempty open real interval of $\mathbb{R}$ and denote $I^* := (I - I) \cap \mathbb{R}_+$, where $\mathbb{R}_+$ stands for the set of nonnegative real numbers. Let $\varphi : I^* \to \mathbb{R}_+$ be a given function. A real valued function $f : I \to \mathbb{R}$ is called *φ-convex* if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + t\varphi((1-t)|x-y|) + (1-t)\varphi(t|x-y|) \tag{1}$$

for all $t \in [0,1]$ and for all $x, y \in I$. If (1) holds for $t = 1/2$ then we say that f is *φ-midconvex.*

In this talk we give some equivalent conditions for φ-convexity. Furthermore, we search relations between the local upper-bounded φ-midconvex functions and φ-convex functions.

Gyula Maksa

University of Debrecen, Hungary
(joint work with **E. Gselmann**)

Nonnegative information functions revisited

Motivated by the known result that there are non-negative information functions different from the Shannon information function, in this talk, we present some properties of the set on which every nonnegative information function coincides with the Shannon's one.

Fruzsina Mészáros

University of Debrecen, Hungary
(joint work with **K. Lajkó**)

Density function solutions of a functional equation

The functional equation

$$f_U(u)\, f_V(v) = f_X\left(\frac{1-v}{1-uv}\right) f_Y(1-uv)\, \frac{v}{1-uv}$$

is investigated for almost all $(u, v) \in (0,1)^2$. Suppose only that the unknown functions $f_X, f_Y, f_U, f_V : (0,1) \to \mathbb{R}$ are density functions of some random variables (i.e. nonnegative and Lebegue integrable with integral 1). Does it follow that they are positive almost everywhere on $(0,1)$?

Using a method of A. Járai in connection with the characterization of the Dirichlet distribution, we give an affirmative answer to this question.

The obtained result is related to an independence property for beta distributions.

Bartosz Micherda

University of Bielsko-Biała, Poland

On the properties of four elements in function spaces

Let X_ρ be a modular space which is a lattice with respect to the ordering $\geq$ given by some pointed convex cone $K \subset X_\rho$. For $x, y \in X_\rho$ denote $x \wedge y = \inf(x,y)$ and $x \vee y = \sup(x,y)$.

Then we say that ρ satisfies *the lower property of four elements* $(LPFE)$ if for any $x, y, w, z \in X_\rho$ such that $x \geq y$, we have

$$\rho(x - w) + \rho(y - z) \geq \rho(x - w \vee z) + \rho(y - w \wedge z),$$

and it satisfies *the upper property of four elements* $(UPFE)$ if for any $x, y, w, z \in X_\rho$ such that $x \geq y$, we have

$$\rho(x - w) + \rho(y - z) \leq \rho(x - w \wedge z) + \rho(y - w \vee z).$$

These inequalities are useful for the study of projection and antiprojection operators in modular spaces (see [1] and [2]).

In our talk we present a class of function modulars which satisfy both (LPFE) and (UPFE). We also give some other examples and counterexamples.

References

[1] G. Isac, G. Lewicki, *On the property of four elements in modular spaces*, Acta Math. Hungar. **83** (1999), 293–301.

[2] B. Micherda, *The properties of four elements in Orlicz-Musielak spaces*, Math. Inequal. Appl. **4** (2001), 599–608.

Vladimir Mityushev

Pedagogical University, Kraków, Poland

Application of functional equations to determination of the effective conductivity of composites with elliptical inclusions

Analysis concerning the transport properties of inhomogeneous materials is of fundamental theoretical interest. Analytical formulae for the macroscopic properties with physical and geometrical parameters in symbolic form is useful to predict the behavior of composites. The method of functional equations is one of the constructive methods to derive such analytical exact and approximate formulae. The present talk is devoted to application of the method to two–dimensional composites with elliptical inclusions. The sizes, the locations and the orientations of the ellipses can be arbitrary. The analytical formulae contains all above geometrical parameters in symbolic form.

Lajos Molnár

University of Debrecen, Hungary

Characterizing some specific elements in spaces of operators and functions and its use

We characterize certain specific elements in spaces of functions or Hilbert space operators and use those characterizations to determine the structures of different kinds of automorphisms and isometries of the underlying spaces.

Janusz Morawiec

Silesian University, Katowice, Poland

(joint work with **R. Kapica**)

Refinement equations and Markov operators

Let $(\Omega, \mathcal{A}, P)$ be a complete probability space, let $L: \Omega \to \mathbb{R}^n$ be a random vector and let $K: \Omega \to \mathbb{R}^{n\times n}$ be a random matrix. We discuss the close connection between the problem of the existence of non-trivial L^1-solutions $f: \mathbb{R}^n \to \mathbb{R}$ of the refinement equation

$$f(x) = \int_\Omega |\det K(\omega)| f(K(\omega)x - L(\omega)) dP(\omega)$$

and the problem of the existence of invariant probability Borel measures of a very special Markov operator defined (on the space of all finite Borel measures on $\mathbb{R}^n$) by

$$M\mu(A) = \int_\Omega \int_{\mathbb{R}^p} \chi_A(K(\omega)^{-1}(x + L(\omega))) d\mu(x) dP(\omega).$$

Jacek Mrowiec

University of Bielsko-Biała, Poland

On stability of some functional equation

Recently, Soon-Mo Jung has proved the Hyers-Ulam stability of the Fibonacci functional equation

$$f(x) = f(x-1) + f(x-2)$$

in the class of functions $f: \mathbb{R} \to X$, where X is a real Banach space. The same method with little modifications may be applied to prove stability of the more general functional equation

$$f(x) = af(x-1) + bf(x-2),$$

where $a, b \in \mathbb{R}$, in the same class of functions. However, for some values of a and b this equation is not stable.

Anna Mureńko

University of Rzeszów, Poland

A generalization of Bernstein-Doetsch theorem

Let V be an open convex subset of a nontrivial real normed space X. We give a partial generalization of Bernstein-Doetsch theorem. Namely, if there exist a base $\mathcal{B}$ of X and a point $x \in V$ such that a midconvex function $f : X \to \mathbb{R}$ is locally bounded above on b-ray at x for each $b \in \mathcal{B}$, then f is convex. Moreover, under the above assumption, f is also continuous in case $X = \mathbb{R}^N$, but not in general.

Adam Najdecki

University of Rzeszów, Poland

On stability of some functional equation

Let S be a nonempty set, $k, n \in \mathbb{N}$ and $g_j : S \times S \to S$ for $j \in \{1, \dots, k\}$. We are going to discuss the stability of the functional equation

$$\sum_{j=1}^{k} f(g_j(s,t)) = f(s)f(t), \quad s, t \in S$$

in the class of functions f from S to the normed algebra $M_n(\mathbb{C})$ of complex $n \times n$ matrices.

Kazimierz Nikodem

University of Bielsko-Biała, Poland

Remarks on strongly convex functions

Let D be a convex subset of a normed space and $c > 0$. A function $f : D \to \mathbb{R}$ is called *strongly convex* with modulus c if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) - ct(1-t)\|x - y\|^2$$

for all $x, y \in D$ and $t \in [0, 1]$. We say that f is *midpoint strongly convex* with modulus c if

$$f\Big(\frac{x+y}{2}\Big) \le \frac{f(x) + f(y)}{2} - \frac{c}{4}\|x - y\|^2, \quad x, y \in D.$$

Some properties of midpoint strongly convex functions (corresponding to the classical results of Jensen convex functions) are presented. A relationship between strong convexity and generalized convexity in the sense of Beckenbach is also given.

Andrey A. Nuyatov
Nizhny Novgorod State University, Russia

Representation of space of entire functions of Fischer's pairs

In [2] resolvability of the equation

$$\psi_1(z)M_{F_1}[f] + ... + \psi_m(z)M_{F_m}[f] = g(z) \quad (1)$$

is proved, $\overrightarrow{\psi} = (\psi_1(z), ..., \psi_m(z)) \in H^m_{C^n}$, $M_{F_j}[f] \equiv (F_j, f(z+w))$ - the operator of convolution in the space $H(C^n)$, which characteristic function is equal to $\varphi_j(z)$, $j = 1, ..., m$. Resolvability of this equation is connected by concept of Fisher's pairs (see [1]):

A pair of polynomials $(P(z), Q(D))$, $D = (D_1, ..., D_n), D_j = \partial/\partial z_j$ forms a *Fischer pair* if

$$H(\mathbf{C^n}) = (P(z)) \oplus \mathrm{Ker}Q(D).$$

In this connection, equation (1) can be written down in the following way

$$\Sigma^0_{k=m} P_k(z) M_{P^*_k}[f] = g(z), \quad (2)$$

where $\mathrm{deg}P_k = \mathrm{deg}P^*_k = k$, $k = 0, ..., m$. Equation (2) will become

$$\Sigma^0_{k=m}(\Sigma^0_{|\alpha|=k} a^k_\alpha z^\alpha)(\Sigma^0_{|\alpha|=k} \overline{a}^k_\alpha D^\alpha f) = g(z). \quad (3)$$

We will show under what conditions the differential equation with variable factors

$$\Sigma^0_{|\beta|=m}[(\Sigma^0_{|\alpha|=m} b_{\alpha\beta} z^\alpha) D^\beta f] = g(z) \quad (4)$$

is led to equation (3), i.e. the factors of equation (3) are expressed through the factors of equation (4). Let $B = ||b_{\alpha\beta}||$ be matrix of factors of equation (4).

Theorem.
If the transposed matrix to B can be represented in the form of $B^T = \Sigma^0_{k=m} B_k$, where $B_k = ||b^k_{\alpha\beta}||$ ($k = m, m-1, ..., 0$) - Hermitean conjugate matrixes of a rank 1, thus the only elements of the last of $\frac{1}{(n-1)!}\sum_{i=0}^k \prod_{j=1}^{n-1}(i+j)$, $n \geq 2$ rows and $\frac{1}{(n-1)!}\sum_{i=0}^k \prod_{j=1}^{n-1}(i+j)$, $n \geq 2$ columns are nonzero, then equation (4) is led to equation (3).

The program which checks conditions of reduction of the given equation to equation (3) and if it is possible is written and expresses the factors of equation (3) through the factors of equation (4) and writes down equation (3).

References

[1] H.S. Shapiro, *An algebraic theorem of E. Fischer, and the holomorphic Goursat problem*, Bull. London Math. Soc. **21** (1989), 513–537.

[2] V.V. Napalkov, *On the theory of linear differential equations with variable coefficients* (Russian), Dokl. Akad. Nauk **397** (2004), 748–750.

Andrzej Olbryś
Silesian University, Katowice, Poland

On some inequality connected with Wright convexity

We consider the functional inequality

$$f(\lambda x + (1-\lambda)y) \leq G(x,y,\lambda)f(x) + [1 - G(x,y,\lambda)]f(y), \qquad x,y \in (a,b),\ \lambda \in (0,1),$$

where $f : (a,b) \to \mathbb{R}$ and $G : (a,b) \times (a,b) \times (0,1) \to \mathbb{R}$ is a function symetric with respect to x and y.

Jolanta Olko
Pedagogical University, Kraków, Poland

On a family of multifunctions

Let $\{f^t, t \in \mathbb{R}\}$, $\{g^t, t \in \mathbb{R}\}$ be groups of increasing homeomorphisms of an interval I such that $f^t \leq g^t$, $t \in \mathbb{R}$.

We study properties of the family $\{H^t, t \in \mathbb{R}\}$ of multifunctions defined as follows $H^t(x) = [f^t(x), g^t(x)]$ for $x \in I$, $t \in \mathbb{R}$.

Zsolt Páles
University of Debrecen, Hungary

An application of Blumberg's theorem in the comparison of weighted quasi-arithmetic means

We present comparison theorems for the weighted quasi-arithmetic means and for weighted Bajraktarević means without supposing in advance that the weights are the same. The results have been obtained jointly with Gyula Maksa under differentiability assumptions. Using Blumberg's theorem (stating, for every real function, the existence of a countable dense set such that the restriction of the function to this set is continuous), these regularity assumptions are completely removed.

Boris Paneah
Technion, Haifa, Israel

Several remarks on approximate solvability of the linear functional equations

We consider the general linear functional operator

$$\mathcal{P}F(x) := \sum_{j=1}^{N} c_j(x) F \circ a_j(x), \quad x \in D \subset \mathbb{R}^p.$$

Here $F \in C(I,B)$ (the space of all B-valued continuous functions on I) with $I = (-1,1)$, B a Banach space, *coefficients* c_j and *arguments* a_j of $\mathcal{P}$ are continuous functions $D \to \mathbb{R}$ and $D \to I$, respectively, D is a domain with a compact closure.

Recently a deep connection between this operator and different problems from analysis, geometry and even gas dynamic has been discovered. In a series of works some existing and uniqueness problems have been studied as well as the overdeterminedness for some types of the operators $\mathcal{P}$ has been established. Because of the linearity of $\mathcal{P}$ studying homogeneous equation $\mathcal{P}F \approx 0$ and, in particular, searching an approximate solution to this equation provokes the special interest (from both theoretical and practical points of view). It worth noting that even the notion of the approximate solution by itself needs to be defined accurately.

At the first part of the talk I formulate and discuss the new notions identifying problem and approximate solution related to linear functional operator $\mathcal{P}$. In particular, it will be clarified the interrelation of the identifying and well-known Ulam problems. It will be explained also that the latter problem bears a direct relation to the approximate solvability rather then to some mythic stability.

At the second part of the talk the set of linear functional operators for which I succeeded in proving the solvability of the identifying problem and the approximate solvability of the equation $\mathcal{P}F \approx 0$ will be described and discussed.

In conclusion a list of the most interesting unsolved problems will be demonstrated.

On approximate solvability of the Cauchy equation of arbitrary degree

The talk is devoted to the well-known but not well studied functional operator

$$\mathfrak{C}_n F := F(0) + \sum_{k=1}^{n} (-1)^k \sum_{1 \le j_1 < \ldots < j_k \le n} F(x_{j_1} + \ldots + x_{j_k}),$$

where $x = (x_1, \ldots, x_n)$ is a point of a bounded domain in $\mathbb{R}^n$ and F is a function: $I \to B$ with B a Banach space and $I = \{t : 0 \le t \le 1\}$. We show at first where from this operator arises in different fields of mathematics and physics, and then we formulate the problem of approximate solvability of the equation $\mathfrak{C}_n F \approx 0$. In the second part of the talk we solve this problem.

Magdalena Piszczek
Pedagogical University, Kraków, Poland

On multivalued iteration semigroups

Let K be a closed convex cone with a nonempty interior in a Banach space and let $G\colon K \to cc(K)$ be a continuous additive multifunction. The equality

$$F_t \circ G = G \circ F_t, \qquad t \ge 0$$

is a necessary and sufficient condition under which the family $\{F_t :\ t \ge 0\}$ of multifunctions

$$F_t(x) = \sum_{i=0}^{\infty} \frac{t^i}{i!} G^i(x), \qquad x \in K,\ t \ge 0$$

is an iteration semigroup.

Dorian Popa
Technical University of Cluj-Napoca, Romania

A property of a functional inclusion connected with Hyers-Ulam stability

We prove that a set-valued map $F : X \to \mathcal{P}_0(Y)$ satisfying the functional inclusion $F(x)\Diamond F(y) \subseteq F(x * y)$ admits, in appropriate conditions, a unique selection $f : X \to Y$ satisfying the functional equation $f(x) \diamond f(y) = f(x * y)$, where $(X, *)$, $(Y, \diamond)$ are square-symmetric grupoids and $\Diamond$ is the extension of $\diamond$ to the collection $\mathcal{P}_0(Y)$ of all nonempty parts of Y.

References

[1] J. Aczél, *Lectures on functional equations and their applications*, Mathematics in Science and Engineering **19**, Academic Press, New York-London, 1966.

[2] J. Brzdęk, A. Pietrzyk, *A note on stability of the general linear equation*, Aequationes Math. **75** (2008), 267–270.

[3] Z. Gajda, R. Ger, *Subadditive multifunctions and Hyers-Ulam stability*, General inequalities **5** (Oberwolfach, 1986), 281–291, Internat. Schriftenreihe Numer. Math. **80** Birkhńuser, Basel, 1987.

[4] D.H. Hyers, G. Isac, Th.M. Rassias, *Stability of functional equations in several variables*, Progress in Nonlinear Differential Equations and their Applications **34**, Birkhäuser Boston, Inc., Boston, MA, 1998.

[5] Zs. Páles, *Hyers-Ulam stability of the Cauchy functional equation on square-symmetric grupoids*, Publ. Math. Debrecen **58** (2001), 651–666.

[6] D. Popa, *A stability result for a general linear inclusion*, Nonlinear Funct. Anal. Appl. **9** (2004), 405–414.

[7] D. Popa, *Functional inclusions on square-symmetric grupoids and Hyers-Ulam stability*, Math. Inequal. Appl. **7** (2004), 419–428.

[8] A. Smajdor, *Additive selections of superadditive set-valued functions*, Aequationes Math. **39** (1990), 121–128.

[9] W. Smajdor, *Subadditive set-valued functions*, Glas. Mat. Ser. III **21 (41)** (1986), 343–348.

[10] W. Smajdor, *Superadditive set-valued functions and Banach-Steinhaus theorem*, Rad. Mat. **3** (1987), 203–214.

Vladimir Yu. Protasov
Moscow State University, Russia

Lipschitz stability of linear operators in Banach spaces

The well-known concept of Ulam-Hyers-Rassias stability for the additive Cauchy equation establishes, in particular, the p-stability of linear maps between Banach spaces for all positive parameters $p \neq 1$. The only exception is the Lipschitz case, when $p = 1$ (see [1] and references therein). One of possible ways to obtain stability results for this case is to introduce the notion of Lipschitz linear stability. Let X, Y be arbitrary Banach spaces and $F : X \to Y$ be a map with the only assumption that there is $K>0$ such that $\|F(x)\| \le K\|x\|$, $x\in X$. For a given $\varepsilon>0$ we consider the following condition on F:

$$\Big\| \{a,b\}_F - \{b,c\}_F \Big\| \le \varepsilon, \qquad a,c\in X,\ b\in [a,c], \tag{1}$$

where $\{\,x_1\,,\,x_2\,\}_F$ denotes the divided difference $\frac{F(x_2)-F(x_1)}{\|\,x_2-x_1\|}$. This condition is fulfilled for $\varepsilon=0$ precisely when F is linear. We say that a map F can be *linearly Lipschitz C-approximated* if there is a linear operator $A: X\to Y$ such that

$$\Big\| \{x_1,x_2\}_{F-A} \Big\| \le C, \qquad x_1,x_2\in X.$$

This means that $\|F(x_1)-F(x_2)-(Ax_1-Ax_2)\| \le C\|x_1-x_2\|$. Observe that if $F(0)=0$, then $\|F(x)-Ax\,\| \le C\|x\|$ for any x. Thus, Lipschitz linear approximation property implies the linear approximation in the sense of Ulam-Hyers-Rassias stability for $p=1$. Consider now the following property called in the sequel Lipschitz linear stability (LLS):

(LLS) For given Banach spaces X and Y there is a function $C(\varepsilon)$, which tends to zero as $\varepsilon\to 0$, such that any map $F: X\to Y$ possessing property (1) can be linearly Lipschitz $C(\varepsilon)$-approximated.

Any Lipschitz ε-perturbation of a linear operator possesses property (1). The question is whether the converse is true: if (1) holds for a map F, then F can be linearly Lipschitz $C(\varepsilon)$-approximated? In other words, if a map $F: X\to Y$ can be linearly Lipschitz ε-approximated on any straight line $l\subset X$, can it be $C(\varepsilon)$-approximated globally on X? This problem was stated for case of functionals (when $Y=\mathbb{R}$) by Prof. Zsolt Páles in 12th ICFEI [2, Problem 2, pp.150–151] both for the entire space X and for convex domains $D\subset X$. First we answer the question of LLS for functionals:

Theorem 1
If X is an arbitrary Banach space and $Y=\mathbb{R}$, then the LLS property holds with $C(\varepsilon)=2\varepsilon$.

The proof is based on the separation principle, and cannot be extended from the case $Y=\mathbb{R}$ to an arbitrary Banach space Y. This extension, nevertheless, can be realized using a totally different idea, which leads to the following result:

Theorem 2
The LLS property holds with $C(\varepsilon)=2\varepsilon$ for any Banach spaces X,Y, whenever X is separable.

It appears that the estimate $C(\varepsilon)=2\varepsilon$ is the best possible in both those theorems, and cannot be improved already for $X=\mathbb{R}^2, Y=\mathbb{R}$. Then we consider LLS for maps F defined on convex open bounded domains $D\subset X$, in which case $C(\varepsilon)$ already depends on the geometry of the domain.

References

[1] Th.M. Rassias, *On the stability of functional equations and a problem of Ulam*, Acta Appl. Math. **62** (2000), 23–130.

[2] *Report of Meeting: 12th ICFEI*, Ann. Acad. Pedagog. Crac. Stud. Math. **7** (2008), 125–159.

Euler binary partition function and refinement equations

Refinement equations, i.e., difference functional equations with the double contractions of the argument have been studied in the literature in great detail due to their applications in functional analysis, wavelets theory, ergodic theory, probability, etc. Any refinement equation is written in the form

$$\varphi(x) = \sum_{k=0}^{d-1} c_k \varphi(2x - k), \tag{1}$$

where $\{c_k\}$ are complex coefficients such that $\sum_{k=0}^{d-1} c_k = 2$. This equation always possesses a unique, up to multiplication by a constant, compactly supported solution φ in the space of distributions $\mathcal{S}'$.

We present a rather surprising application of refinement equations to a well-known problem of the combinatorial number theory: the asymptotics of the Euler partition function. For an arbitrary integer $d \geq 2$ the binary partition function $b(k) = b(d,k)$ is defined on the set of nonnegative integers k as the total number of different binary expansions $k = \sum_{j=0}^{\infty} d_j 2^j$, where the "digits" d_j take values from the set $\{0, \dots, d-1\}$. The asymptotic behavior of $b(k)$ as $k \to \infty$ was studied by L. Euler, K. Mahler, N.G. de Bruijn, D.E. Knuth, B. Reznick and others.

It appears that the exponent of growth of the function $b(k)$ can be expressed by the solution φ of refinement equation (1) with equal coefficients $c_k = \frac{1}{d}$. Using this argument we answer two open questions formulated by B. Reznick in 1990 (see [1]).

References

[1] B. Reznick, *Some binary partition functions*, Analytic number theory (Allerton Park, IL, 1989), 451–477, Progr. Math. **85**, Birkhäuser Boston, Boston, MA, 1990.

[2] V.Yu. Protasov, *On the problem of the asymptotics of the partition function*, Math. Notes **76** (2004), 144–149.

Viorel Radu

West University of Timişoara, Romania

Ulam-Hyers stability of functional equations in locally convex probabilistic spaces: a fixed point method

In [1] and [2] some generalized Ulam-Hyers stability results for Cauchy functional equation have been proved. Our aim is to outline the results concerning the generalized Ulam-Hyers stability for different other kinds of functional equations.

The fixed point method (cf. [4]) will be emphasized, for functions defined on linear spaces and taking values in fuzzy normed spaces and locally convex probabilistic spaces.

References

[1] D. Miheţ, V. Radu, *On the stability of the additive Cauchy functional equation in random normed spaces*, J. Math. Anal. Appl. **343** (2008), 567–572.

[2] L. Cădariu, V. Radu, *On the stability of the Cauchy functional equation: a fixed points approach*, Iteration theory (ECIT'02), 43–52, Grazer Math. Ber. **346**, Karl-Franzens-Univ. Graz, Graz, 2004.

[3] D.H. Hyers, G. Isac, Th.M. Rassias, *Stability of functional equations in several variables*, Progress in Nonlinear Differential Equations and their Applications **34**, Birkhäuser Boston, Inc., Boston, MA, 1998.

[4] V. Radu, *The fixed point alternative and the stability of functional equations*, Fixed Point Theory **4** (2003), 91–96.

Ewa Rak
University of Rzeszów, Poland
(joint work with **J. Drewniak**)

Domination and distributivity inequalities

Domination is a property of operations which plays an important role in considerations connected with the distributivity functional inequalities. Schweizer and Sklar [4] introduced the notion of domination for associative binary operations with common range and common neutral element. In particular, the property of domination was considered in the families of triangular norms and conorms (see e.g. [1, 2, 3]). In our considerations we shall show some of dependencies between the property of domination and the subdistributivity or the superdistributivity of operations on the unit interval.

References

[1] J. Drewniak, P. Drygaś, U. Dudziak, *Domination between multiplace operations*, Issues in Soft Computing. Decisions and Operations Research, EXIT, Warszawa 2005, 149–160.

[2] S. Saminger-Platz, *The dominance relation in some families of continuous Archimedean t-norms and copulas*, Fuzzy Sets and Systems **160** (2009), 2017–2031.

[3] P. Sarkoci, *Domination in the families of Frank and Hamacher t-norms*, Kybernetica (Prague) **41** (2005), 349–360.

[4] B. Schweizer, A. Sklar, *Probabilistic metric spaces*, North-Holland Series in Probability and Applied Mathematics, North-Holland Publishing Co., New York, 1983.

Themistocles M. Rassias
National Technical University of Athens, Greece

Stanisław Marcin Ulam

In this special session, I will talk briefly on the life and works of S.M.Ulam.

Maciej Sablik

Silesian University, Katowice, Poland

Bisymmetrical functionals

Let $\Omega_i, i = 1, 2$ be compact sets. Consider spaces $B(\Omega_i, \mathbb{R})$ of bounded functions defined on Ω_i, and let F and G be functionals defined in $B(\Omega_1, \mathbb{R})$ and $B(\Omega_2, \mathbb{R})$, respectively. We characterize F and G such that the equation

$$G\left(F_t(x(s,t))\right) = F\left(G_s(x(s,t))\right)$$

holds for every $x \in B(\Omega_1 \times \Omega_2, \mathbb{R})$, under some additional regularity assumptions. It turns out that F and G are conjugated to an integral with respect to some Radon measure in B_i. The main tool in the proof is a result of Gy. Maksa from [1].

Reference

[1] Gy. Maksa, *Solution of generalized bisymmetry type equations without surjectivity assumptions*, Aequationes Math. **57** (1999), 50–74.

Ekaterina Shulman

Vologda State Pedagogical University, Russia

(joint work with **F. Nazarov**)

Stable quasi-mixing of the horocycle flow

We consider the behavior of a one-parameter subgroup of a Lie group under the influence of a sequence of kicks. Our approach follows [1] where a special case of the problem was related to an asymptotic behavior of "approximate" solutions of some functional equations on a discrete group.

Let a Lie group G act on a set X, and $(h^t)_{t\in\mathbb{R}}$ be a one-parameter subgroup of G; it is a dynamical system acting on X. We perturb this system by a sequence of kicks $\{\phi_i\} \subset G$. The kicks arrive with some positive period τ. The dynamics of the kicked system is described by a sequence of products $P_\tau(i) = \phi_i h^\tau \phi_{i-1} h^\tau \dots \phi_1 h^\tau$ that depend on the period τ.

A dynamical property of a subgroup (h^t) is called *kick stable*, if for every sequence of kicks $\{\phi_i\}$, the kicked sequence $P_\tau(i)$ inherits this property for a "large" set of periods τ. The property we will concentrate on, is quasi-mixing.

A sequence $\{P(i)\}$ acting on a measure space (X, μ) by measure-preserving automorphisms is called *quasi-mixing* if there exists a subsequence $\{i_k\} \to \infty$ such that for any two L_2-functions F_1 and F_2 on X

$$\int_X F_1(P(i_k)x)F_2(x)d\mu \to \int_X F_1(x)d\mu \int_X F_2(x)d\mu \qquad \text{when} \quad k \to \infty.$$

In our case $X = \mathrm{PSL}(2,\mathbb{R})/\Gamma$, where $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ is a lattice. The group $\mathrm{PSL}(2,\mathbb{R})$ acts on X by left multiplication. The principal tool used in [1] for the study of stable mixing in this setting, is the Howe-Moore theorem which gives the geometric description of quasi-mixing systems: if the sequence $P(i)$ is unbounded then it is quasi-mixing.

It follows from the Howe-Moore theorem that the horocycle flow $h^t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ is quasi-mixing on X. We prove that it is kick stably quasi-mixing. This answers the question raised by L. Polterovich and Z. Rudnick in [1].

Let us mention an application to second order difference equations. A discrete Schrödinger-type equation is the equation

$$q_{k+1} - (2 + tc_k)q_k + q_{k-1} = 0, \qquad k \geq 1. \tag{1}$$

COROLLARY.
For every sequence $\{c_n\}$, the set of the parameters $t \in \mathbb{R}_+$ for which all solutions of the difference equation (1) are bounded, has finite measure.

Reference

[1] L. Polterovich, Z. Rudnick, *Kick stability in groups and dynamical systems*, Nonlinearity **14** (2001), 1331–1363.

Justyna Sikorska
Silesian University, Katowice, Poland

A direct method for proving the Hyers-Ulam stability of some functional equations

We study the stability of the equation of the form

$$f(x) = af(h(x)) + bf(-h(x))$$

with some conditions imposed on constants a, b and function h. The results are later applied (by use of a direct method - the Hyers sequences) for proving the stability of several functional equations.

Barbara Sobek
University of Rzeszów, Poland

Quadratic equation of Pexider type on a restricted domain

Let X be a real (or complex) locally convex linear topological space. Assume that U is a nonempty, open and connected subset of $X \times X$. Let

$$U_1 := \{x : (x, y) \in U \text{ for some } y \in X\},$$

$$U_2 := \{y : (x, y) \in U \text{ for some } x \in X\},$$

$$U_+ := \{x + y : (x, y) \in U\}$$

and

$$U_- := \{x - y : (x, y) \in U\}.$$

We consider the functional equation

$$f(x+y)+g(x-y)=h(x)+k(y), \qquad (x,y)\in U,$$

where $f: U_+ \to Y$, $g: U_- \to Y$, $h: U_1 \to Y$ and $k: U_2 \to Y$ are unknown functions and $(Y,+)$ is a commutative group. The general solution of the equation is given. We also present an extension result.

Joanna Szczawińska
Pedagogical University, Kraków, Poland

Some remarks on a family of multifunctions

Let $f: \mathbb{R} \to \mathbb{R}$ denote the function given by

$$f(t)=\sum_{n=0}^{\infty} a_n t^n, \qquad t\in\mathbb{R},$$

where $a_n \geq 0$ for $n\in\mathbb{N}$. If K is a closed convex cone in a real Banach space and $H: K \to \mathrm{cc}(K)$ a linear and continuous set-valued function with nonempty, convex and compact values in K, then for all $t\geq 0$, the set-valued function

$$F^t(x):=\sum_{n=0}^{\infty} a_n t^n H^n(x), \qquad x\in K$$

is linear and continuous and

$$F^t\circ F^s(x)\subseteq \sum_{n=0}^{\infty} c_n H^n(x), \qquad x\in K,$$

where

$$c_n=\sum_{k=0}^{n} a_k a_{n-k} t^k s^{n-k}, \qquad t,s\geq 0.$$

The necessary and sufficient condition for the equality

$$F^t\circ F^s(x)=\sum_{n=0}^{\infty} c_n H^n(x), \qquad x\in K,\ t,s\geq 0$$

will be given.

Tomasz Szostok
Silesian University, Katowice, Poland

On a functional equation stemming from some property of triangles

Basing on some geometrical property discovered by G. Monge, in [1] authors considered the following functional equation

$$\left|\frac{1}{2}(y-x)f\left(\frac{x+y}{2}\right)-\frac{1}{2}(f(y)-f(x))\frac{x+y}{2}\right|=\int_x^y f(t)dt+\frac{1}{2}xf(x)-\frac{1}{2}yf(y).$$

They proved that the only solutions of this equation are the affine functions. Roughly speaking this means that Monge theorem works only for collinear points.

In the present talk we modify this equation in such way that it will be satisfied by some functions different from $f(x)=ax+b$. Then we solve the obtained equation.

Reference

[1] C. Alsina, M. Sablik, J. Sikorska, *On a functional equation based upon a result of Gaspard Monge*, J. Geom. **85** (2006), 1–6.

Jacek Tabor
Jagiellonian University, Kraków, Poland
(joint work with **Józef Tabor** and **M. Żołdak**)

Approximate (ε,p)-midconvexity for $p\in[0,1]$

For $p\in[0,1]$ we put

$$T_p(x):=\sum_{k=0}^{\infty}\frac{1}{2^k}d^p(2^kx),\qquad x\in\mathbb{R},$$

where $d(x)=2\mathrm{dist}(x,\mathbb{Z})$ and by 0^0 we understand 0.

A function $f:I\to\mathbb{R}$, where I is a subinterval of $\mathbb{R}$, is called (ε,p)-*midconvex* if

$$\mathcal{J}f(x,y):=\frac{f(x)+f(y)}{2}-\frac{f(x)+f(y)}{2}\le\varepsilon|x-y|^p,\qquad x,y\in I.$$

It is known that if f is a continuous (ε,p)-midconvex function, then

$$f(rx+(1-r)y)-rf(x)-(1-r)f(y)\le\varepsilon T_p(r|x-y|),\qquad x,y\in I,r\in[0,1].$$

The above estimation is optimal for $p=0$ (theorem of C.T. Ng and K. Nikodem) and $p=1$ (theorem of Z. Boros). Zs. Palés asked what happens in the case when $p\in[0,1]$.

We show that the above problem can be reduced to verification of the following hypotheses:

$$\min\{\mathcal{J}d^p(x,y)+\frac{1}{2}d^p(x-y),\mathcal{J}d^p(x,y)+\frac{1}{2}\mathcal{J}d^p(2x,2y)+\frac{1}{4}d^p(2x-2y)\}\le d^p(\frac{x-y}{2})$$

for $x,y\in[-1,1]$. The above inequality can be easily verified for $p=0$ and $p=1$ (giving in particular another proof of the result of Z. Boros). Although numerical simulations support the assertion that the above hypothesis holds for all $p\in(0,1)$, we were not able to prove it.

Józef Tabor
University of Rzeszów, Poland
(joint work with **Jacek Tabor** and **A. Mureńko**)

Jensen semiconcave functions with power moduli

We study the relation between Jensen semiconcavity and semiconcavity in the case when modulus of semiconcavity is of the form $\omega(r) = Cr^p$, $p \in (0, 1]$. As it is known continuous Jensen semiconcave function with modulus ω is semiconcave with modulus

$$\tilde{\omega}(r) := \sum_{k=0}^{\infty} \omega\left(\frac{r}{2^k}\right).$$

In case of $\omega(r) = Cr^p$, $p \in (0, 1]$ we improve this result and determine the smallest $\tilde{\omega}$.

Gheorghe Toader
Technical University of Cluj-Napoca, Romania
(joint work with **S. Toader**)

Invariance in some families of means

As it is known from the classical example of the arithmetic-geometric mean of Gauss (see [1]), the determination of a $(M, N)-$invariant mean P is a very difficult problem. That is why we study the (equivalent) problem of finding a mean N which is complementary to M with respect to P. For the determination of complementaries, three methods have been used: the direct calculation (see [4]), the use of the methods of functional equations (see [2]), and the series expansion of means (see [3]). In the current paper we consider the method of series expansion of means to study the invariance in the family of extended logarithmic means.

References

[1] J.M. Borwein, P.B. Borwein, *Pi and the AGM. A study in analytic number theory and computational complexity*, Canadian Mathematical Society Series of Monographs and Advanced Texts, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1987.

[2] Z. Daróczy, Zs. Páles, *Gauss-composition of means and the solution of the Matkowski-Sutô problem*, Publ. Math. Debrecen **61** (2002), 157–218.

[3] D.H. Lehmer, *On the compounding of certain means*, J. Math. Anal. Appl. **36** (1971), 183–200.

[4] Gh. Toader, S. Toader, *Greek means and the arithmetic-geometric mean*, RGMIA Monographs, Victoria University, 2005 (http://rgmia.vu.edu.au/monographs).

Peter Volkmann
University, Karlsruhe, Germany, and Silesian University, Katowice, Poland

Continuity of solutions of a certain functional equation

The continuous solutions $f : \mathbb{R} \to \mathbb{R}$ of the functional equation

$$\min\{f(x+y), f(x-y)\} = |f(x) - f(y)|$$

had been given in a talk during the Conference on Inequalities and Applications at Noszvaj 2007 (http://riesz.math.klte.hu/∼cia07). Here we show that the continuity of a solution of this functional equation follows from the continuity at one point.

Marek C. Zdun
Pedagogical University, Kraków, Poland

Iteration groups and semigroups – recent results

This is a survey talk on selected topics concerning iteration groups and semigroups where some progress has been achieved during the last years. Especially we concern on the problem of embeddability of given functions in iteration groups and iterative roots.

Marek Żołdak
University of Rzeszów, Poland
(joint work with **Jacek Tabor** and **Józef Tabor**)

Bernstein-Doetsch type theorem for approximately convex functions

Let X be a real topological vector space, let D be a subset of X and let $\alpha : X \to [0, \infty)$ be an even function locally bounded at zero.

A function $f : D \to \mathbb{R}$ is called (α, t)-*preconvex* (where $t \in (0, 1)$ is fixed), if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \alpha(x-y)$$

for all $x, y \in D$ such that $[x, y] \subset D$.

We give a version of Bernstein-Doetsch theorem and some related results for such functions.