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Problems and Remarks

1. Remark. *On application of the multiplication formula to $\frac{1}{n}$-stable probability distributions*

From the multiplication formula for the gamma function we obtain for $n \in \mathbb{N}$, $x > 0$, that

$$\frac{n\Gamma(nx)}{\Gamma(x)} = (n^n)^x \cdot \frac{\Gamma(x+\frac{1}{n})}{\Gamma(\frac{1}{n})} \cdot \frac{\Gamma(x+\frac{2}{n})}{\Gamma(\frac{2}{n})} \cdot \ldots \cdot \frac{\Gamma(x+\frac{n-1}{n})}{\Gamma(\frac{n-1}{n})}. \tag{1}$$

If $\xi_{\frac{1}{n}}, \xi_{\frac{2}{n}}, \ldots, \xi_{\frac{n-1}{n}}$, are independent random variables with $\Gamma(\frac{1}{n}, 1), \ldots, \Gamma(\frac{n-1}{n}, 1)$ probability distributions, respectively, then for the right hand side we can write

$$RHS_{(1)}(n,x) = E(n^n \cdot \xi_{\frac{1}{n}} \cdot \xi_{\frac{2}{n}} \cdot \ldots \cdot \xi_{\frac{n-1}{n}})^x, \tag{2}$$

where E stands for the expectation.

On the other hand we can write

$$LHS_{(1)}(n,x) = E(\sigma_{1/n})^{-x}, \tag{3}$$

if $\sigma_{1/n}$ has the strictly stable probability distribution defined by its Laplace transform

$$E\left(e^{-s\sigma_{1/n}}\right) = e^{(-s^{\frac{1}{n}})}, \qquad \operatorname{Re} s \geq 0.$$

Indeed, by Fubini's theorem we obtain

$$\begin{aligned} E(\sigma_{1/n})^{-x} &= \int_0^\infty v^{-x} dP_{\sigma_{1/n}}(v) \\ &= \int_0^\infty \left(\int_0^\infty y^{x-1} e^{-vy} dy\right) \frac{1}{\Gamma(x)} dP_{\sigma_{1/n}}(v) \\ &= \frac{1}{\Gamma(x)} \int_0^\infty y^{x-1} e^{-y^{1/n}} dy \\ &= n\frac{\Gamma(nx)}{\Gamma(x)}, \qquad x > 0. \end{aligned}$$

Thus, by the uniqueness theorem for the inverse two-sided Laplace transform, from (1) – (3) we obtain the equality of distributions

$$\sigma_{1/n} \stackrel{d}{=} \frac{1}{n^n \cdot \xi_{1/n} \cdot \xi_{2/n} \cdot \ldots \cdot \xi_{(n-1)/n}}, \qquad n = 1, 2, 3, \ldots. \tag{4}$$

For $n = 2$ it is known as a property of the $\Gamma(\frac{1}{2}, 1)$ probability distribution (P. Lévy).

Joachim Domsta

2. Problem. *Lipschitz perturbation of continuous linear functionals*

Let X be a normed space, $D \subseteq X$ be an open convex set and let $f\colon D \to \mathbb{R}$ be a Lipschitz perturbation of a linear functional, i.e., let f be of the form

$$f = x^* + \ell,$$

where x^* is a continuous linear functional and $\ell\colon D \to \mathbb{R}$ is an ε-Lipschitz function, i.e.,

$$|\ell(x) - \ell(y)| \le \varepsilon \|x - y\| \qquad (x, y \in D).$$

Then, for $x, y \in D$ and $t \in [0,1]$, we have

$$\begin{aligned} &|tf(x) + (1-t)f(y) - f(tx + (1-t)y)| \\ &\qquad = |t\ell(x) + (1-t)\ell(y) - \ell(tx + (1-t)y)| \\ &\qquad \le t|\ell(x) - \ell(tx + (1-t)y)| + (1-t)|\ell(y) - \ell(tx + (1-t)y)| \\ &\qquad \le t\varepsilon\|x - (tx + (1-t)y)\| + (1-t)\varepsilon\|y - (tx + (1-t)y)\| \\ &\qquad = 2\varepsilon t(1-t)\|x - y\|. \end{aligned}$$

On the other hand, in the case $X = \mathbb{R}$, we have the following converse of the above observation.

Claim

Let I be an open interval and $\varepsilon \ge 0$. Assume that $f\colon I \to \mathbb{R}$ satisfies, for all $x, y \in I$ and $t \in [0,1]$, the inequality

$$|tf(x) + (1-t)f(y) - f(tx + (1-t)y)| \le 2\varepsilon t(1-t)|x - y|. \tag{1}$$

Then there exists a constant $c \in \mathbb{R}$ such that the function $\ell\colon I \to \mathbb{R}$ defined by $\ell(x) := f(x) - cx$ is (2ε)-Lipschitz.

The proof is elementary and is left to the reader. However, the following more general and open problem seems to be of interest.

Problem

Does there exist a constant γ (that may depend on X and D) such that, whenever a function $f\colon D \to X$ satisfies inequality (1) for all $x, y \in D$ and $t \in [0,1]$, then there exists a continuous linear functional x^* such that the function $\ell := f - x^*$ is $\gamma\varepsilon$-Lipschitz on D?

Zsolt Páles

3. Problems. 1. Find all mappings $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that the following functional equation is satisfied:

$$\|T\vec{u} \times T\vec{v}\| = \|\vec{u} \times \vec{v}\|, \qquad \text{for all } \vec{u}, \vec{v} \in \mathbb{R}^3.$$

Geometrically the problem is asking for the determination of all mappings $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ which preserve area of parallelograms in the Euclidean 3-dimensional space.

2. Find all mappings $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that the following functional equation is satisfied

$$|(T\vec{u} \times T\vec{v}) \cdot T\vec{w}| = |(\vec{u} \times \vec{v}) \cdot \vec{w}|, \qquad \text{for all } \vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3.$$

Geometrically the problem is asking for the determination of all mappings $T\colon \mathbb{R}^3 \to \mathbb{R}^3$, which preserve volume of parallelepipeds in the Euclidean 3-dimensional space.

Note. In the above two problems the symbols $\times, \cdot$ denote vector product and dot (scalar) product, respectively.

Remark. It will be interesting to formulate and solve the analogous functional equations for mappings $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ for the determination of all mappings which preserve area (resp. volume) of the surfaces of balls (resp. solid balls) in the Euclidean 3-dimensional space. The same problem remains open for ellipsoids in $\mathbb{R}^3$.

3. Examine whether there exists a mapping $f\colon B \to \mathbb{R}^3$, that is at the same time *harmonic* as well as *homeomorphism.*

Remark. $f\colon B \to \mathbb{R}^3$ is a harmonic mapping if its three coordinate real-valued functions on the unit ball B are harmonic functions, i.e., if each one of the three coordinate functions of f satisfies the Laplace equation in B. The mapping $f\colon B \to \mathbb{R}^3$ is a homeomorphism if it is a bijective and bicontinuous mapping.

The same problem remains open for the n-dimensional case where $n = 4, 5, \ldots$.

Themistocles M. Rassias

4. Remark. *Some remarks on the talk of T. Miura*

Recently the number of papers whose title includes the word combination "Ulam stability" threateningly grows. Little by little these works begin to involve operators which are very far from the linear functional operators $\mathcal{P}$ in several variables (in the framework of these operators this notion have appeared). The interest to the type of stability in question is easily explained by its evident connection with the very important problem of the approximate solvability of the inequality $\|\mathcal{P}F\| < \varepsilon$. But to the late days any progress in the solvability of this problem in the class of functional equations in several variables was connected mainly with a success in guessing new types of the operators $\mathcal{P}$ to which it is possible (after a series of substitutions and arithmetic transformations) to apply the original construction of Hyers.

As to the classical operators of analysis (integral, differential, partial differential, etc.) to which from time to time some authors turn in order to cross them with the Ulam stability, for these operators the "Ulam problem" is successfully solving under different names during 70 years in the framework of functional analysis (Banach, Riesz, Schauder, Leray and others). I'll demonstrate this on the basis of the Miura's talk "A note on stability of Volterra type integral equation". The speaker delivered the following result.

Let $f\colon \mathbb{R} \to B$ be a $C(\mathbb{R}, B)$-function, B a Banach space and

$$(Tf)(t) = f(0) + \alpha(t) \int_0^t p(s,t) f(s) ds$$

with α and p being continuous maps to $\mathbb{C}$. Then there is a function $\varphi\colon \mathbb{R} \to B$ depending on f such that $\varphi - T\varphi = 0$ and $\|f - \varphi\|_C < m\varepsilon$ if $\|f - Tf\| \le \varepsilon$ with m a constant depending neither on f nor ε.

(As a matter of fact, the speaker dealt with the simplest case $\alpha = 1$, $p(s,t) = p(s)$.) From the point of view of functional analysis this is a standard exercise related to the invertibility of linear operators in B-space. No hint at stability! The following solution does not require any comment. Denote by E the identical operators in $C(I, B)$ with I – a compact interval in $\mathbb{R}$, and let $f \in C(I, B)$. Then

1. the operator T is compact in $C(I, B) \Rightarrow$
2. the range of the operator $E - T$ is closed $\Rightarrow$

3. the *a priori* estimate

$$\inf_{\varphi\in\ker(E-T)} |f-\varphi|_{C(I,B)} \le m|(E-T)f|_{C(I,B)}, \qquad f\in C(I,B)$$

holds.

This completes the solution.

In the case $\alpha = 1$, $p(s,t) = p(s)$ the space $\ker(E-T)$ is one-dimensional and consists of the functions $\varphi = \lambda \exp(\int_0^t p(s)ds)$ with $\lambda \in B$.

In my opinion, the majority of results related to integro-differential operators with the reference to the stability, as a matter of fact, has the same nature: some usual property of an inverse operator is treated as the Ulam stability. But from this point of view any classical result in the theory of boundary problems for partial differential equations (the unique solvability of the Dirichlet problem for the Laplace operator, for example) may be treated as Ulam stability.

Boris Paneah

5. Remark. *Some remarks on the talk of Z. Kominek*

In his talk Prof. Z. Kominek considered the operator

$$\mathcal{P}: f(t) \to f(x+2y) + f(x) - 2f(x+y) - 2f(y)$$

from $C(\mathbb{R})$ to $C(\mathbb{R}^2)$ and formulated the following proposition: there is a function $w(x,y)$ such that if $|(\mathcal{P}f)(x,y)| < |w(x,y)|$ for all points $(x,y)\in\mathbb{R}^2$ then the equation $\mathcal{P}F = 0$ is uniquely solvable, and for some function ψ the relation $|f-F| \le \psi(w)$ holds. No information about w and ψ had been mentioned.

It is easily seen that the above operator $\mathcal{P}: C(I) \to C(D)$ with $D = \{(x,y) \mid x+2y \le 1,\ x\ge 0,\ y \ge 0\}$ and $I = [0,1]$ satisfies all conditions formulated in my talk and providing solvability of the identifying problem for $\mathcal{P}$. According to the main result of this talk, if $|(\mathcal{P}f)(x,y)|_{\langle 2\rangle} < \varepsilon$ for all points (x,y) of the straight line $\Gamma = \{(x,y) \mid x = \frac{1}{3}t,\ y\frac{1}{3}t;\ \ 0\le t\le 1\}$, then the inequality $|f(t) - \lambda t^2|_{\langle 2\rangle} < c\varepsilon$ holds for a constant λ and all points t, $0\le t\le 1$. The constant c does not depend on f nor t, $|\cdot|_{\langle 2\rangle}$ is the norm in the space $C_{\langle 2\rangle}$ of continuous in I functions satisfying the 2-Hölder condition at $t=0$. What is important here is that the initial condition of the smallness of $\mathcal{P}f$ is imposed only at points of an one-dimensional manifold Γ and the approximate solution f of the relation $|\mathcal{P}_\Gamma f| < \varepsilon$ is close not to the subspace $\ker\mathcal{P}$, but to the subspace $\ker\mathcal{P}_\Gamma = \{\varphi \mid \varphi(3t) - 2\varphi(2t) - \varphi(t) = 0,\ \ 0\le t\le 1\}$, where $\mathcal{P}_\Gamma$ denotes the restriction of the operator $\mathcal{P}$ to Γ.

Boris Paneah

6. Remark. *On functional equations "of Kuczma's type"*

The first paper on functional equations written by Marek Kuczma (1935–1991) had appeared 50 years ago. Together with his colleagues and students he developed the theory of functional equations called "in a single variable" or "iterative" — later on.

Having this anniversary in mind I proposed to introduce in the title of our paper [1] the name "functional equation of Kuczma's type".

But, motivated by what have been said at the Conference on names assigned to stability problems, I have found this idea was not good. First of all, the late Marek Kuczma himself would be against it. And all who knew him personally would confirm this prediction. Moreover, the new name is unprecise, may led to confusions, and the existing ones are satisfactory.

The aim of this remark is to declare that we decided to change the title of our paper, as indicated in [1].

[1] B. Choczewski, M. Czerni, *Special solutions of a linear functional equation of Kuczma's type.* New title: *Special solutions of a linear iterative functional equation*, Aequationes Math., to appear.

Bogdan Choczewski

7. Problem. *A functional equation with two complex variables*

The functional equation

$$\varphi(z+2\pi i)=\varphi(z), \qquad z\in\mathbb{C} \tag{1}$$

in the class of entire functions has the general solution of the form $\varphi(z)=\psi(\exp z)$, where ψ is an arbitrary entire function. Equation (1) characterizes the complex exponent.

Let $f(z)$ be a given polynomial or entire function. Consider now the functional equation

$$\varphi[z+2\pi i, f(z)]=\varphi[z, f(z)], \qquad z\in\mathbb{C}, \tag{2}$$

where $\varphi(z,w)$ is unknown entire function with respect to z and to w.

Conjecture

The general solution of (2) *has the form* $\varphi(z,w)=\psi(\exp z,w)$*, where* ψ *is an arbitrary entire function of two variables.*

It is worth noting that an artificial insert of the exponent in φ does not solve the problem. For instance, the function $\psi_0(u,w)=\ln u$ produces $\varphi_0(z,w)=\ln\exp z=z$ in the strip $0\le \operatorname{Im} z\le 2\pi$ periodically continued onto $\mathbb{C}$. The function φ_0 satisfies (2), however, φ_0 and ψ_0 are not entire functions.

One can see also that the functional equation $\varphi(z+2\pi i,w)=\varphi(z,w)$, $(z,w)\in\mathbb{C}^2$ (compare to equation (1)) has only exponent in z solutions. But the restriction $w=f(z)$ in (2) yields complications.

The case $f(z)=z$ and its application to Arnold's problem [1, p.168–170] of topologically elementary functions were discussed in [2].

[1] V. I. Arnold (ed.), Arnold's Problems, Springer, Berlin, 2004.

[2] V. Mityushev, *Exponent in one of the variables*, Jan Długosz University of Częstochowa, Scientific Issues, Mathematics XII, Częstochowa, 2007.

Vladimir Mityushev

8. Problems. *Stability of the orthogonality preserving property and related problems*

1. As it was reminded in Prof. Aleksej Turnšek's talk, the orthogonality preserving property for linear mappings between Hilbert spaces is stable. Namely (cf. [1], [4]), if $f\colon X\to Y$ is a *linear* mapping satisfying

$$x\perp y \implies fx\perp^{\varepsilon} fy, \qquad x,y\in X$$

(where $u\perp^{\varepsilon} v$ means that $|\langle u|v\rangle|\le\varepsilon\|u\|\,\|v\|$), then there exists a *linear* mapping $g\colon X\to Y$ satisfying

$$x\perp y \implies gx\perp gy, \qquad x,y\in X$$

and such that

$$\|fx - gx\| \le \left(1 - \sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\right) \cdot \min\{\|fx\|, \|gx\|\}, \qquad x \in X.$$

Problem 1. The question is whether the linearity can be omitted in the above statement (both in assumption and in assertion).

2. Orthogonality preserving property can also be defined for mappings between normed spaces, with one of the possible notions of orthogonality. Attempting to solve the stability problem for this property, with respect to the *isosceles orthogonality* ($u \perp v \Leftrightarrow \|u+v\| = \|u-v\|$) I encountered the following problem concerning the stability of isometries.

Problem 2. Let $f\colon X \to Y$ be a linear mapping between Banach spaces satisfying

$$\big|\, \|fx - fy\| - \|x - y\| \,\big| \le \varepsilon \|x - y\|, \qquad x, y \in X.$$

Does there exists a linear isometry $I\colon X \to Y$ such that

$$\|fx - Ix\| \le \delta(\varepsilon)\|x\|, \qquad x \in X$$

(with some $\delta : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\lim_{\varepsilon \to 0^+} \delta(\varepsilon) = 0$)?

Without the linearity assumption, the question has a negative answer, as it was shown by G. Dolinar [3, Proposition 4].

Yet during the meeting Problem 2 has been solved by Prof. Vladimir Protasov. For finite-dimensional spaces the answer to the question is positive (some compactness argument is sufficient) whereas for infinite ones it is generally not true. Namely, for an increasing sequence of positive numbers $\{\alpha_k\}_{k\in\mathbb{N}}$ such that the series $\sum_{k\in\mathbb{N}}(1-\alpha_k^2)$ converges, consider the following norm in a Hilbert space l_2:

$$\|x\|_\alpha := \sup\left\{ \|x\|_{l_2}, \left|\frac{x_1}{\alpha_1}\right|, \left|\frac{x_2}{\alpha_2}\right|, \ldots \right\}, \qquad x = (x_1, x_2, \ldots) \in l_2.$$

Denote the space l_2 endowed with the new norm $\|\cdot\|_\alpha$ by H_α (this space is reflexive). Now, let $A_k\colon H_\alpha \to H_\alpha$ be an operator interchanging the kth and $(k+1)$st coordinates. It can be shown that A_k is an approximate isometry (with given ε provided that k is sufficiently big) but it cannot be approximated by a linear isometry, as the only linear isometries on the considered space are coordinate symmetries (i.e., $Te_i = \pm e_i$, $i = 1, 2, \ldots$).

Afterwards, A. Turnšek pointed out that the problem has been already considered in the literature, e.g. in [2].

[1] J. Chmieliński, *Stability of the orthogonality preserving property in finite-dimensional inner product spaces*, J. Math. Anal. Appl. **318** (2006), 433–443.

[2] G.G. Ding, *The approximation problem of almost isometric operators by isometric operators*, Acta Math. Sci. (English Ed.), **8** (1988), 361–372.

[3] G. Dolinar, *Generalized stability of isometries*, J. Math. Anal. Appl. **242** (2000), 39–56.

[4] A. Turnšek, *On mappings approximately preserving orthogonality*, J. Math. Anal. Appl. **336** (2007), 625–631.

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