Arithmetic neighbourhoods of numbers: an extended summary

Apoloniusz Tyszka

Let $\boldsymbol{K}$ be a ring and let A be a subset of $\boldsymbol{K}$. We say that a map $f : A \to \boldsymbol{K}$ is *arithmetic* if it satisfies the following conditions: if $1 \in A$ then $f(1) = 1$, if $a, b \in A$ and $a+b \in A$ then $f(a+b) = f(a) + f(b)$, if $a, b \in A$ and $a \cdot b \in A$ then $f(a \cdot b) = f(a) \cdot f(b)$. We call an element $r \in \boldsymbol{K}$ *arithmetically fixed* if there is a finite set $A \subseteq \boldsymbol{K}$ (an *arithmetic neighbourhood* of r inside $\boldsymbol{K}$) with $r \in A$ such that each arithmetic map $f : A \to \boldsymbol{K}$ fixes r, i.e. $f(r) = r$.

All previous articles on arithmetic neighbourhoods ([2], [1], [3]) dealt with a description of a situation where for an element in a field there exists an arithmetic neighbourhood. If $\boldsymbol{K}$ is a field, then any $r \in \boldsymbol{K}$ is arithmetically fixed if and only if $\{r\}$ is existentially first-order definable in the language of rings without parameters ([3]). Therefore, presentation of the arithmetic neighbourhood of the element r belonging to the field $\boldsymbol{K}$ is the simplest way of expression that $\{r\}$ is existentially definable in $\boldsymbol{K}$.

Let $\widetilde{\boldsymbol{K}}_n$ ($n = 1, 2, 3, ...$) denote the set of all $r \in \boldsymbol{K}$ for which there exists an arithmetic neighbourhood A of r such that $\mathrm{card}(A) \le n$. For a positive integer n we define the set of equations E_n by

$$E_n = \{x_i = 1 : \ 1 \le i \le n\} \cup$$
$$\{x_i + x_j = x_k : \ 1 \le i \le j \le n, \ 1 \le k \le n\} \cup \{x_i \cdot x_j = x_k : \ 1 \le i \le j \le n, \ 1 \le k \le n\}$$

Conjecture. If a non-empty subset of E_n forms a system of equations that is consistent in $\mathbb{R}$ ($\mathbb{C}$), then this system has a solution being a sequence of real numbers (complex numbers) whose absolute values are not greater than $2^{2^{n-2}}$.

For $n = 1$ estimation by $2^{2^{n-2}}$ can be replaced by estimation by 1. For $n > 1$ estimation by $2^{2^{n-2}}$ is the best estimation. Indeed, let $n > 1$ and $\widetilde{x_1} = 1$, $\widetilde{x_2} = 2^{2^0}$, $\widetilde{x_3} = 2^{2^1}$, ..., $\widetilde{x_n} = 2^{2^{n-2}}$. In any ring $\boldsymbol{K}$ of characteristic 0, from the system of all equations belonging to E_n and are fulfilled by $\widetilde{x_1}$, ..., $\widetilde{x_n}$, it follows that $x_1 = \widetilde{x_1}$, ..., $x_n = \widetilde{x_n}$.

By Theorem 3 in [2] $\widetilde{\mathbb{R}}_n \subseteq \mathbb{R}^{\mathrm{alg}} = \{x \in \mathbb{R} : x \text{ is algebraic over } \mathbb{Q}\}$. By this, the Conjecture implies

$$\widetilde{\mathbb{R}}_n \subseteq \mathbb{R}^{\mathrm{alg}} \cap [-2^{2^{n-2}}, \ 2^{2^{n-2}}]$$

By Corollary 2 in [2] $\widetilde{\mathbb{C}}_n \subseteq \mathbb{Q}$. By this, the Conjecture implies

$$\widetilde{\mathbb{C}}_n \subseteq \mathbb{Q} \cap [-2^{2^{n-2}}, \ 2^{2^{n-2}}]$$

Let w denote the unique real root of the polynomial $x^3 - x^2 - x - 3$. Let $n \in \mathbb{Z}$, $n \ge 3$, $S_n = \{1, 10, 20, 30\} \cup \{3, 3^2, 3^3, ..., 3^n\}$, $S = \bigcup_{n=3}^{\infty} S_n$, $B_n = \{1, 5, 25, 26\} \cup \{3, 3^2, 3^3, ..., 3^n\}$, $B = \bigcup_{n=3}^{\infty} B_n$.

Theorem 1. There is an arithmetic map $\gamma : S \to \mathbb{Z}[\sqrt{-1}]$ which moves all $r \in S \setminus \{1\}$. For each $r \in S_n \setminus \{1\}$ we have: S_n is an arithmetic neighbourhood of r inside $\mathbb{R}$, and so too inside $\mathbb{Q}$ and $\mathbb{Z}$, S_n is not an arithmetic neighbourhood of r inside $\mathbb{Z}[\sqrt{-1}]$.

Theorem 2. There is an arithmetic map $\phi : B \to \mathbb{Q}$ which moves all $r \in B \setminus \{1\}$. For each $r \in B_n \setminus \{1, 5\}$ we have: B_n is an arithmetic neighbourhood of r inside $\mathbb{Z}$, B_n is not an arithmetic neighbourhood of r inside $\mathbb{Q}$.

Theorem 3. There is an arithmetic map $\psi : \{-4\} \cup B \to \mathbb{Q}(w)$ which moves all $r \in \{-4\} \cup B \setminus \{1\}$. For each $r \in \{-4\} \cup B_n \setminus \{1\}$ we have: $\{-4\} \cup B_n$ is an arithmetic neighbourhood of r inside $\mathbb{Q}$, $\{-4\} \cup B_n$ is not an arithmetic neighbourhood of r inside $\mathbb{Q}(w)$.

Theorem 4. If $\boldsymbol{K} = \mathbb{Q}(\sqrt{5})$ or $\boldsymbol{K} = \mathbb{Q}(\sqrt{33})$, then for infinitely many rational numbers r for some arithmetic neighbourhood of r inside $\mathbb{Q}$ this neighbourhood is not a neighbourhood of r inside $\boldsymbol{K}$.

Let n be an integer, and assume that $n \geq 3$ and $n \notin \{2^2, 2^3, 2^4, ...\}$. We find the smallest integer $\rho(n)$ such that $n^3 \leq 2^{\rho(n)}$. Then $2^{\rho(n)}$ has four digits in the number system with base n. Let

$$2^{\rho(n)} = m_3 \cdot n^3 + m_2 \cdot n^2 + m_1 \cdot n + m_0$$

where $m_3 \in \{1, 2, ..., n-1\}$ and $m_2, m_1, m_0 \in \{0, 1, 2, ..., n-1\}$. Let

$$\begin{aligned} \mathcal{J}(n) = \left\{-1,\ 0,\ 1,\ -\frac{1}{2},\ -\frac{1}{2^2},\ -\frac{1}{2^3},\ ...,\ -\frac{1}{2^{\rho(n)}},\ n,\ n^2\right\} \cup \\ \left\{k \cdot n^3 : \quad k \in \{1, 2, ..., m_3\}\right\} \cup \\ \left\{m_3 \cdot n^3 + k \cdot n^2 : \quad k \in \{1, 2, ..., m_2\}\right\} \cup \\ \left\{m_3 \cdot n^3 + m_2 \cdot n^2 + k \cdot n : \quad k \in \{1, 2, ..., m_1\}\right\} \cup \\ \left\{m_3 \cdot n^3 + m_2 \cdot n^2 + m_1 \cdot n + k : \quad k \in \{1, 2, ..., m_0\}\right\} \end{aligned}$$

Theorem 5. $\mathcal{J}(n)$ is an arithmetic neighbourhood of n inside $\mathbb{R}$, and so too inside $\mathbb{Q}$. $\mathcal{J}(n)$ is not an arithmetic neighbourhood of n inside $\mathbb{C}$.

For which integers r (rational numbers r) each arithmetic neighbourhood of r inside $\mathbb{Z}$ (inside $\mathbb{Q}$) is also a neighbourhood of r inside each ring extending $\mathbb{Z}$ (extending $\mathbb{Q}$)? For which integers r each arithmetic neighbourhood of r inside $\mathbb{Z}$ is also a neighbourhood of r inside $\mathbb{Q}$?

For $r = 1$, $r = 0$, $r = 2$, $r = \frac{1}{2}$, and for any ring $\boldsymbol{K}$ with $r \in \boldsymbol{K}$, each arithmetic neighbourhood of r inside $\boldsymbol{K}$ is also a neighbourhood of r inside each ring extending $\boldsymbol{K}$, so for the numbers $r = 1$, $r = 0$, $r = 2$, $r = \frac{1}{2}$ we have positive answers.

The full text is available at
http://www.cyf-kr.edu.pl/~rttyszka/neighbourhoods.pdf
http://arxiv.org/abs/math.NT/0602310

References

[1] G. Lettl, *Finitely arithmetically fixed elements of a field*, 9 pages, Arch. Math. (Basel), to appear. Presented at 70th Workshop on General Algebra, Institute of Discrete Mathematics and Geometry, Vienna University of Technology, May 26–29, 2005.

[2] A. Tyszka, *A discrete form of the theorem that each field endomorphism of* $\mathbb{R}$ ($\mathbb{Q}_p$) *is the identity*, Aequationes Math. 71 (2006), no. 1–2, 100–108.

[3] A. Tyszka, *On* $\emptyset$*-definable elements in a field*, Collect. Math. 58 (2007), no. 1, 73–84, http://www.imub.ub.es/collect/last.html

Apoloniusz Tyszka
Technical Faculty
Hugo Kołłątaj University
Balicka 116B, 30-149 Kraków, Poland
E-mail address: *rttyszka@cyf-kr.edu.pl*