On Applications of Functional Equations in Physics

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Point of interest

Nowadays, problems in Physics are generally modelled by partial differential equations. Before the development of the differential calculus, the physical processes were often analyzed in terms of functions. There was an input variable x (or several input variables) and an output variable $f(x)$, and $f(x)$ had to satisfy some relations corresponding to some properties of the physical process which were often known by observation. This leads to some functional equation for f.

Functional equations represent an alternative way of modelling problems in Physics. The interest of modelling physical problems by functional equations is that we do not have to assume the differentiability of the function f. Consequently, the functional equations lead often to other solutions than those given by partial differential equations, and these other solutions can be of interest to physicists.

In this talk, I will present some applications of functional equations in Physics. Through these examples, I will explain how the functional equations appear in the physical problem, how we use it to solve the problem, and I will talk about the solutions which are not used for solving the problem, but which can be of interest.

First example: (1) $\Phi(\alpha x) = \beta\, \Phi(x) \quad (x \in I)$

$I = (0, +\infty)$ or $[0, +\infty)$ or $\mathbb{R}$, $\alpha, \beta \in \mathbb{R}$, $\alpha > 1$, $\beta > 0$

- **First application of (1)**

There has been a controversy about the fact that Descartes could have got the gravity law before Galileo.

In 1891, Paul Tannery, an historian of sciences, tried to explain in his book: *Mémoires Scientifiques, Tome VI*, how Descartes made an error when he thought he had obtained the gravity law. The argument of Descartes written in a letter of 1629 was the following:

"If the body runs a distance $2x$, it runs the first distance x during the time $3t$ and the second distance during the time t."

Writing $x = f(t)$, P. Tannery explains that the argument of Descartes leads to the functional equation:

$$(D)\ \ f(4t) = 2f(3t) \qquad \Longleftrightarrow \qquad f(\frac{4}{3}t) = 2f(t) \qquad (t \geq 0)$$

Then, P. Tannery deduced without explanation: $f(t) = k\ t^{\frac{\ln 2}{\ln 4 - \ln 3}}$ and he concluded that Descartes hadn't got the gravity law.

So, he was thinking that power functions were the only solutions of (D). However, the study of (1) proves that it is not true. Namely, we have the following result (cf. N. Brillouët-Belluot, J. Gapaillard, *On a simple linear iterative functional equation*, Demonstratio Mathematica XXXI 4, 1998, 735-752):

Theorem.

1. If $\Phi : (0, +\infty) \to \mathbb{R}$ is a solution of (1) and if $\lim\limits_{x \to 0+} \dfrac{\Phi(x)}{x^c}$ exists and is finite with $c = \dfrac{\ln \beta}{\ln \alpha}$, then $\Phi(x) = k\, x^c \quad (x > 0)$.

2. The general C^n-solution $\Phi : (0, +\infty) \to \mathbb{R}$ of (1) is given by:

$$\Phi(x) = \beta^p\, \varphi_0(\alpha^{-p} x) \quad (x > 0)$$

with $p = [\dfrac{\ln x}{\ln \alpha}]$ and $\varphi_0 : [1, \alpha) \to \mathbb{R}$ arbitrary with some further condition.

3. Let $q = [\dfrac{\ln \beta}{\ln \alpha}]$. The general C^n-solution $\Phi : [0, +\infty) \to \mathbb{R}$ of (1) is given by:

- if $\beta < 1$, $\Phi \equiv 0$
- if $\beta = 1$, Φ is an arbitrary constant function
- if $\beta \neq \alpha^q$, when $n > q$ $\Phi \equiv 0$
  when $n \leq q$, $\Phi(0) = 0$ and $\Phi_{|\mathbb{R}_+}$ is a C^n-solution of (1)
- if $\beta = \alpha^q$, when $n \geq q$ $\Phi(x) = k\, x^q$
  when $n < q$, $\Phi(0) = 0$ and $\Phi_{|\mathbb{R}_+}$ is a C^n-solution of (1)

So, if $n \leq 2$, all C^n-solutions $f : [0, +\infty) \to \mathbb{R}$ of (D) depend on an arbitrary function, and if $n > 2$, $f \equiv 0$. Therefore, the argument of P. Tannery was not right since the power functions are not the only solutions of (D). However, his assertion that Descartes hadn't got the gravity law was true since the functions kt^2 are not solutions of (D). And so, really, Descartes hadn't got the gravity law.

On the other hand, if we consider the right functional equation modelling the gravity law:

$$(G) \quad f(2t) = 4f(t) \quad (t \geq 0)$$

the only C^n-solutions of (G) with $n \geq 2$ are: $f(t) = k\,t^2$, which gives the gravity law!

We have to mention that this was not the argument of Galileo (which appeared in 1630), since he proved directly that the distance run by the body was proportional to the square of the time. However, the experiments and the argument of Galileo lead to another functional equation:

$$\frac{x((n+1)t) - x(nt)}{x(nt) - x((n-1)t)} = \frac{2n+1}{2n-1} \qquad (t > 0, n \in \mathbb{N}) \qquad \text{with} \quad x(0) = 0.$$

Galileo proved that this implies: $x(t) = k\,t^2$.

• **Second application of (1): Problem of evaporation in presence of wind.**

In Meteorology, in the problem of evaporation in the presence of wind, the evaporative Reynolds number π_1 satisfies approximately the relation: $\pi_1 = g(\pi_2)\,\pi_3$ where π_2 is the advective Reynolds number, π_3 is the saturation deficit, and g is a function which satisfies the functional equation:

$$g(k\pi_2) = k^\gamma\ g(\pi_2), \quad \text{where } k, \gamma \text{ are positive constants}$$

$$\text{and} \quad \lim_{x\to 0+} \frac{g(x)}{x^\gamma} \quad \text{exists.}$$

The previous theorem implies: $g(\pi_2) = \beta\ \pi_2^\gamma$ where β is an arbitrary positive constant.

Second example: D'Alembert's functional equation

(2) $\quad f(x+y) + f(x-y) = 2f(x)f(y) \quad (x, y \in \mathbb{R})$

• **First application of (2): The composition of forces in Mechanics**

J. Aczél, *Lectures on Functional Equations and their Applications*, 1966

J. Aczél, J. Dhombres, *Functional Equations in Several Variables*, 1989

• **Second application of (2): The problem of a vibrating streched string by D'Alembert (1747)**

D'Alembert considered this problem under the hypothesis of small vibrations. He denotes by $y(t,s)$ the ordinate of a point of the string of abscissa s, $0 \leq s \leq \ell$, at the moment t. The end points of the string are fixed: $y(t,0) = y(t,\ell) = 0$, and the string is rectilinear at the beginning: $y(0,s)=0$.

The partial differential equation modelling this problem is:

$$\frac{\partial^2 y}{\partial t^2} = a^2 \ \frac{\partial^2 y}{\partial s^2} \qquad (\ a=1 \text{ in the case of d'Alembert})$$

D'Alembert didn't write explicitely this equation. He used the total differentials and wrote: $dy = p\,dt \ + \ q\,ds$. Then, he deduced successively:

$$dy = \frac{\phi(t+s)+\Delta(t-s)}{2}\ dt + \frac{\phi(t+s)-\Delta(t-s)}{2}\ ds$$

and

$$y = \psi(t+s) \ + \ \Gamma(t-s)$$

Because of the hypothesis that the string is rectilinear at the time $t=0$, he got:

$$y = \psi(s+t) \ - \ \psi(s-t)$$

Then, he considered the following condition for the initial velocity:

$$\Big(\frac{\partial y}{\partial t}\Big)_{t=0} = 2\psi'(s) = A\sin(\lambda s)$$

and he obtained the final expression for $y(t,s)$:

$$y(t,s) = -\frac{A}{4} \quad [(c^{i\frac{n}{\ell}(t+s)} + c^{-i\frac{n}{\ell}(t+s)}) - (c^{i\frac{n}{\ell}(t-s)} + c^{-i\frac{n}{\ell}(t-s)})]$$

D'Alembert observed that the function ψ satisfies: $\psi(t+s) - \psi(t-s) = \Delta(t)\,\Gamma(s)$ and he asserted without explanation that only $\psi(x) = A\ \cos(\lambda x)$ could satisfy this equation.

The functional equation:

$$(2A) \qquad f(x+y) - f(x-y) = 2g(x)\ h(y) \qquad (x,y\in\mathbb{R})$$

is a special case of Wilson's second generalization of d'Alembert's functional equation:

$$(2W) \qquad f(x+y) + g(x-y) = h(x)\ k(y) \qquad (x,y\in\mathbb{R})$$

The functional equation $(2W)$ has been first considered by Wilson in 1920. He gave some properties of the solutions, but he didn't solve the equation completely.

From the set of continuous solutions of d'Alembert's equation, we may find all continuous solutions $f,g,h:\mathbb{R}\to\mathbb{R}$ of $(2W)$ (cf. J. Aczél, *Vorlesungen über Funktionalgleichungen und ihre Anwendungen*, 1961).

The, we can deduce the nontrivial continuous solutions $f,g,h:\mathbb{R}\to\mathbb{R}$ of $(2A)$ (cf. H. Stetkaer, *Wilson's functional equations on groups*, Aeq. Math. 49, 252-275, 1995):

either $\begin{cases} g(x) = & A_1 \cos \mu x + A_2 \sin \mu x \\ h(x) = & B \ \sin \mu x \\ f(x) = & -A_2 B \ \cos \mu x + A_1 B \ \sin \mu x + C \end{cases}$

or $\begin{cases} g(x) = & A_1 + A_2 cx \\ h(x) = & cx \\ f(x) = & A_1 cx + \frac{1}{2} A_2 (cx)^2 + C \end{cases}$

where A_1, A_2, B, C, μ are complex numbers,... and for d'Alembert only $f(x) = A \ \cos \mu x \ \ (\mu \in \mathbb{R})$ could be a continuous solution of $(2A)$!

The problem of a vibrating string corresponds to the one-dimensional wave equation. So, we may think that the general wave equation could similarly be stated in terms of functional equations.

Third example: **Gołąb-Schinzel equation**

$$(3) \qquad f(x + f(x)y) = f(x) \ f(y) \quad (x, y \in \mathbb{R})$$

or

$$(4) \qquad F(x + s(x)y) = s(x) \ F(y) \quad (x, y \in \mathbb{R})$$

Applications of (4) appeared in: P. Kahlig, J. Matkowski, *A modified Gołąb-Schinzel equation on a restricted domain (with applications to Meteorology and Fluid Mechanics)*, Sitzungsber. Abt II (2002) 211, 117-136.

The continuous solutions of (4) are given by: $\begin{cases} F(x) = & F(0) \ s(x) \\ s(x + s(x)y) = & s(x) \ s(y) \end{cases}$

from which we get: $s(x) =$ either $1 + cx$ or $\mathrm{Sup}(0, 1 + cx)$ $(x \in \mathbb{R})$

- **First application of (4): Relation between degrees centigrades x and degrees Fahrenheit $F(x)$**

The solution $F(x) = \mathrm{Sup}\,(32 + \dfrac{9}{5}x, 0)$ of (4) leads to the "traditionalistic" version (we avoid negative temperature values),
the other solution $F(x) = 32 + \dfrac{9}{5}x$ leads to the "modernistic" version.

- **Second application of (4): Evaporation of cloud droplets**

The evaporation of a spherical cloud droplet of radius $r(t)$ in a dry environment may be modelled by the following differential equation:

$$(5) \qquad \begin{cases} r^2\ (\rho \dfrac{dr}{dt} + c) = 0 & (t \geq 0) \\ r(0) = R \end{cases}$$

where ρ is the density and c some positive constant.
By solving this differential equation, we get the solution: $r(t) = \mathrm{Sup}(R - \dfrac{c}{\rho} t, 0)$.

The differential equation (5) is invariant under the transformation:

$$t \to \sigma(\tau)\, t + \tau, \qquad r \to \sigma(\tau)\, r \qquad \text{with} \qquad \tau \geq 0,\ \sigma(\tau) > 0$$

Therefore, any solution r of (5) satisfies: $r(\sigma(\tau)\, t + \tau) = \sigma(\tau)\ r(t)$, which is the pexiderized Gołąb-Schinzel equation (4) on $[0, +\infty)$.

The solution $\begin{cases} \sigma(\tau) = & \mathrm{Sup}(1 - k\,\tau, 0) \\ r(t) = & r(0)\ \sigma(t) = R\ \sigma(t) \end{cases}$ with $k = \dfrac{c}{\rho R}$ leads to the previous solution without solving the differential equation (5).

Remark: Why to choose this transformation? This is the only differentiable transformation under which the differential equation (5) is invariant.

• Third application of (4): Water discharging from a reservoir

A cylindric tank contains water up to the height $h(t)$ above an orifice at the time t. The outflow velocity $v(t)$ is, according to Toricelli's law: $v(t) = \sqrt{2\,g\,h(t)}$, where g is the acceleration of gravity. The continuity of liquid mass implies:

$$(6) \qquad \frac{dh}{dt} = -0.6\,\frac{A_0}{A}\,\sqrt{2\,g\,h(t)} = -k\,\sqrt{h(t)} \qquad (t \geq 0) \quad \text{with} \quad h(0) = H$$

where A, A_0 are the cross sections of the tank and of the orifice respectively (the minus sign indicates falling water level).

Integrating this differential equation, we get:

$$h(t) = H \quad (\mathrm{Sup}(1 - \frac{k}{2\sqrt{H}}\ t, 0))^2 \qquad (t \geq 0)$$

The differential equation (6) is invariant under the transformation:

$$t \to \sqrt{\sigma(\tau)}\, t + \tau, \quad r \to \sigma(\tau)\, r \qquad \text{with} \qquad \tau \geq 0,\ \sigma(\tau) > 0$$

Therefore, any solution h of (6) satisfies: $h(\sqrt{\sigma(\tau)}\, t + \tau) = \sigma(\tau)\ h(t) \qquad (t \geq 0)$

which is equivalent to: $\begin{cases} h(t) = h(0)\ \sigma(t) = H\ \sigma(t) \\ \sigma(\sqrt{\sigma(\tau)}\ t + \tau) = \sigma(\tau)\ \sigma(t) \end{cases}$

or
$$\begin{cases} h(t) = H\ (\varphi(t))^2 \\ \varphi(\varphi(\tau)\ t + \tau) = \varphi(\tau)\ \varphi(t) \\ \varphi(t) = \sqrt{\sigma(t)} \end{cases}$$

By using the continuous solutions of the Gołąb-Schinzel equation on $[0, +\infty)$, we get the solution: $\varphi(\tau) = \mathrm{Sup}(1 - C\,\tau, 0)$ with $C = \dfrac{k}{2\sqrt{H}}$. This leads to the previous solution without solving the differential equation (6).

Remark concerning the construction of a clepsydra (water clock).

For the construction of a clepsydra where the water level varies uniformly with respect to the time, we have: $h(t) = -B\,t + H$ and therefore

$$\frac{dh}{dt} = -B = -0.6\,\frac{A_0}{A}\,\sqrt{2\,g\,h(t)}$$

So, if we suppose that the cross sections of the tank are disks and if r is the tank radius at height h, we get: $h = b^2\,\dfrac{A^2}{A_0^2} = K\,r^4$, which means that the water clock should have a cup-like shape. Still existing Egyptian water clocks come close to this requirement.

Conclusion.

We see through these examples that, in physical problems, functional equations may appear in different ways:

1. the functional equations appear naturally from the properties of the physical process, like for the gravity law ;

2. the physical process is stated in terms of some partial differential equations. We can solve these equations and the solutions satisfy some functional equations which eventually lead to some other solutions ;

3. the physical process is stated in terms of some partial differential equations which are invariant under some transformation of the variables. This leads to some functional equations which give, among the solutions, the required solution of the physical process. With this method, we don't solve the differential equations, but the related functional equations, whose solutions are not necessarily differentiable.

This way of modelling physical processes by functional equations might be of interest both to physicists and to mathematicians. It seems that the most appropriate field in Physics where this method can be applied is Fluid Mechanics. It could be of interest for functional equationists to investigate this topic.