



## Mirosław Adamek

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### On generalized affine functions

Let  $\mathcal{F}$  be a family of real functions defined on a nonempty interval  $I \subset \mathbb{R}$ . We say that  $\mathcal{F}$  is a *two-parameter family* on  $I$  if, for any two different points  $x_1, x_2 \in I$  and for any  $t_1, t_2 \in \mathbb{R}$ , there exists exactly one  $\varphi \in \mathcal{F}$  such that

$$\varphi(x_i) = t_i \quad \text{for } i = 1, 2.$$

The unique function  $\varphi \in \mathcal{F}$  determined by the points  $x_1, x_2 \in I$  and values  $t_1, t_2 \in \mathbb{R}$  will be denoted by  $\varphi_{(x_1, t_1)(x_2, t_2)}$ .

A function  $f: I \rightarrow \mathbb{R}$  is called  $\mathcal{F}$ -*affine* if for any different points  $x_1, x_2 \in I$  and  $t \in [0, 1]$  the following equality holds

$$f(tx_1 + (1-t)x_2) = \varphi_{(x_1, f(x_1))(x_2, f(x_2))}(tx_1 + (1-t)x_2).$$

If function  $f$  satisfies the above equality only with a fixed  $t \in (0, 1)$ , then such function we call  $(\mathcal{F}, t)$ -*affine*.

In the talk we will present some general properties about them and also Kuchn-type theorem for  $(\mathcal{F}, t)$ -affine functions.

## Roman Badora

Silesian University, Katowice, Poland

### On the stability of the nonlinear functional equation

In the talk we present an elementary proof of the following generalization of Baker's theorem (J.A. Baker, *The stability of certain functional equations*, Proc. Amer. Math. Soc. **112** (1991), 729–732) on the stability of the nonlinear functional equation.

**THEOREM.**

Let  $S$  be a nonempty set and let  $(X, d)$  be a complete metric space. Assume that  $f: S \rightarrow S$  and the function  $F: S \times X \rightarrow X$  satisfies

$$d(F(t, x), F(t, y)) \leq \lambda(t)d(x, y), \quad t \in S, x, y \in X,$$

where  $\lambda: S \rightarrow \mathbb{R}$ . Suppose that  $\phi: S \rightarrow X$  satisfies

$$d(\phi(t), F(t, \phi(f(t)))) \leq \varepsilon(t), \quad t \in S,$$

where  $\varepsilon: S \rightarrow \mathbb{R}$  and

$$\sum_{n=2}^{\infty} \varepsilon(f^{n-1}(t)) \prod_{i=0}^{n-2} \lambda(f^i(t)) < +\infty, \quad t \in S.$$

Then there exists a unique function  $\Phi: S \rightarrow X$  such that

$$\Phi(t) = F(t, \Phi(f(t))), \quad t \in S$$

and

$$d(\Phi(t), \phi(t)) \leq \varepsilon(t) + \sum_{n=2}^{\infty} \varepsilon(f^{n-1}(t)) \prod_{i=0}^{n-2} \lambda(f^i(t)), \quad t \in S.$$

**Anna Bahyrycz**

Pedagogical University, Kraków, Poland

## Forti's example of an unstable homomorphism equation

We present a proof of some property of the function introduced by G.L. Forti, which is used to show that the homomorphism equation for some group is not stable.

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**Karol Baron**

Silesian University, Katowice, Poland

## On linear iterative equations for distribution functions

Given a probability space  $(\Omega, \mathcal{A}, P)$  and a function  $\tau: \mathbb{R} \times \Omega \rightarrow \mathbb{R}$  we consider the equation

$$F(x) = \int_{\Omega} F(\tau(x, \omega)) P(d\omega)$$

in some classes of distribution functions. We get results on existence, uniqueness and convergence of successive approximations.

## Bogdan Batko

Pedagogical University, Kraków, Poland

### Superstability of some alternative Cauchy functional equations

We are going to discuss the superstability of some alternative functional equations. Consider, for instance, Mikusinski's functional equation

$$f(x+y) \cdot (f(x+y) - f(x) - f(y)) = 0, \quad x, y \in G \quad (1)$$

in the class of complex functions defined on an abelian group  $G$ . We ask if every unbounded solution of the inequality

$$|f(x+y) \cdot (f(x+y) - f(x) - f(y))| \leq \varepsilon, \quad x, y \in G,$$

with given  $\varepsilon > 0$ , must be an exact solution of Mikusinski's equation (1).

## Walter Benz

University of Hamburg, Hamburg, Germany

### A conditional functional inequality

Let  $X$  be a real inner product space of finite or infinite dimension  $\geq 3$ , and  $t$  be a fixed element of  $X$  with  $t^2 = 1$ . Put  $H = \{h \in X \mid ht = 0\}$  and observe that to every  $x \in X$  there exists uniquely determined  $\bar{x} \in H$  and  $x_0 \in \mathbb{R}$  such that  $x = \bar{x} + x_0t$ . If  $x, y \in X$ ,  $x \leq y$  is defined by  $\|\bar{y} - \bar{x}\| \leq y_0 - x_0$ . We are interested in special solutions  $f: X \rightarrow X$ , so-called casual automorphisms, of the conditional functional inequality

$$\forall x, y \in X \quad x \leq y \Rightarrow f(x) \leq f(y),$$

generalizing a theorem of A.D. Alexandrov, V.V. Ovchinnikova and E.C. Zeeman. (See my book "Classical Geometries in Modern Contexts", Birkhäuser, Basel-Boston-Berlin, 2005.)

## Elena V. Blinova

University of Nizhny Novgorod, Nizhny Novgorod, Russia

### On $\Omega$ -explosions in smooth skew products of interval maps

Joint work with L.S. Efremova.

This talk is devoted to the research of  $C^0$ - and  $C^1$ -  $\Omega$ -explosions in  $C^1$ -smooth skew products of interval maps, i.e., maps of the type

$$F(x, y) = (f(x), g_x(y)), \quad (1)$$

where  $(x, y) \in I$ ,  $I$  is the closed rectangle in the plane, in the additional assumption on the closure of the set  $\text{Per}(F)$  of  $F$ - periodic points.

DEFINITION.

Let  $z_1 = (x, y_1), z_2 = (x, y_2)$  be periodic points of  $F$ . The point  $z_1$  is called the accessible from  $z_2$  ( $z_2 \rightarrow^a z_1$ ) if for every  $\varepsilon > 0$  there is an  $\varepsilon$ -chain from  $z_2$  to  $z_1$  by the map  $F$  restricted to  $\bigcup_{y \in \text{Orb}(x)} \{y\} \times I$ , where  $\text{Orb}(x)$  is  $f$ - periodic orbit of  $x$ .

The criterion of  $C^0$ - $\Omega$ -explosion for a map (1) is given in the terms of the properties of the set  $\text{Per}(F)$ .

**THEOREM A**

$C^1$ -smooth map  $F$  with the closed set of periodic points permits a  $C^0$ - $\Omega$ -explosion if and only if  $F$  satisfies one of the following conditions:

1. the set  $\text{Per}(F)$  is connected, and there exists at least one point  $x \in \text{Per}(f)$  such that the set  $\text{Per}(\tilde{g}_x)$  is not connected, here  $\tilde{g}_x = g_{f^{n-1}(x)} \circ \dots \circ g_{f(x)} \circ g_x$ ,  $n$  is the least period of  $x$ ;
2. the set  $\text{Per}(F)$  is not connected (let  $K_i$  be the connected components of  $\text{Per}(F)$ ), and either one of the connected components satisfies the condition (1) or there exists a finite number of connected components  $K_i$ ,  $i = 1, 2, \dots, m$  of  $\text{Per}(F)$  such that for all  $i = 1, 2, \dots, m$ ,  $K_i \xrightarrow{a} K_{i+1}$ , where  $K_{m+1} = K_1$ .

**THEOREM B**

If the set  $\text{Per}(F)$  of  $C^1$ -smooth map (1) is closed, then there is  $\varepsilon > 0$  such that for every map  $F^* \in B_\varepsilon^1(F)$   $\text{Per}(F^*)$  is a closed set, here  $B_\varepsilon^1(F)$  is  $\varepsilon$ -neighborhood of  $F$  in the space of  $C^1$ -smooth skew products of interval maps with  $C^1$ -norm.

The next theorem is the main result in the research of  $C^1$ - $\Omega$ -explosions in  $C^1$ -smooth maps (1).

**THEOREM C**

$C^1$ -smooth map (1) with a closed set of periodic points doesn't permit  $C^1$ - $\Omega$ -explosions.

Finally, we present the example of the one-parameter family of  $C^1$ -smooth maps of type (1) with the closed set of periodic points, where one can observe the new type of bifurcations: the periodic orbit with the period 2 and the periodic orbit with the period 4 appear at once from the fixed point.

This research is partially supported by RFBR, grant No 04-01-00457.

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**Nicole Brillouët-Belluot**

Ecole Centrale de Nantes, Nantes, France

On applications of functional equations in physics

Nowadays, problems in Physics are generally modelled by partial differential equations. Before the development of the differential calculus, the physical processes were often described by functional equations.

Functional equations represent an alternative way of modelling problems in Physics.

In this talk, I will present some applications of functional equations in Physics. Through these examples, I will explain how the functional equations appear in the physical process and what can be the interest of modelling physical problems by functional equations.

## Bruno Brive

University of Science and Technology, Lille, France

### Differential equations of infinite order

Joint work with Prof. Atzmon (Tel Aviv University, Israel).

We consider the following functional equation

$$\sum_{n \geq 0} a_n \frac{d^n}{dz^n} f(z) = g(z) \quad (*)$$

where  $f$  and  $g$  are functions of a complex variable  $z$  and  $a_n$  are complex numbers. This is an inhomogeneous linear differential equation of (possibly) infinite order with constant coefficients. Examples of such equations are given by linear difference–differential equations with constant coefficients. Many authors have studied equations (\*) in various contexts: Nörlund, Valiron, Malgrange, Martineau, Guelfond, . . . . Many results are given in Berenstein and Gay, *Complex Analysis and Special Topics in Harmonic Analysis*, Springer, 1995.

We look at (\*) from a functional analysis viewpoint. We consider the equation (\*) when  $f$  and  $g$  belong to a weighted  $L^p$  space of entire functions. Under general assumptions, the differentiation operator  $D = \frac{d}{dz}$  is bounded on this space. We define the left-hand side of (\*) by Riesz holomorphic functional calculus.

We give a necessary and sufficient condition for the operator  $\sum_{n \geq 0} a_n D^n$  to be surjective. In this case, it admits moreover a bounded linear right-inverse.

We also give an application to the particular equation

$$f(z+1) - f(z) = g(z)$$

which was first investigated by Guichard and Hurwitz.

## Janusz Brzdęk

Pedagogical University, Kraków, Poland

### On stability of the linear functional equation

Joint work with Dorian Popa and Bing Xu.

Let  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ ,  $X$  be a Banach space over  $\mathbb{K}$ ,  $S$  be a nonempty set,  $f: S \rightarrow S$ ,  $F: S \rightarrow X$ ,  $m$  be a positive integer, and  $a_j: S \rightarrow \mathbb{K}$  for  $j = 1, \dots, m$ . We present some results concerning stability of the general linear functional equation (in single variable)

$$\varphi(f^m(x)) = a_1(x)\varphi(f^{m-1}(x)) + \dots + a_{m-1}(x)\varphi(f(x)) + a_m(x)\varphi(x) + F(x), \quad (1)$$

where  $\varphi: S \rightarrow X$  is the unknown function and  $f^p$  denotes the  $p$ -th iterate of  $f$ , i.e.,  $f^0(x) = x$  and  $f^{p+1}(x) = f(f^p(x))$  for  $p = 0, 1, 2, \dots$

For instance, in the special case where all the functions  $a_1, \dots, a_p$  are constant we have the following result.

THEOREM.

Suppose that  $r_1, \dots, r_m \in \mathbb{K}$  are the roots of the characteristic equation

$$r^m - a_1 r^{m-1} - \dots - a_{m-1} r - a_m = 0,$$

$\delta > 0$ , and one of the following two conditions holds.

- (i)  $|r_j| > 1$  for every  $j = 1, \dots, m$ .
- (ii)  $f$  is bijective and  $|r_j| \neq 1$  for every  $j = 1, \dots, m$ .

If a function  $\varphi_s: S \rightarrow X$  satisfies

$$\|\varphi_s(f^m(x)) - \sum_{i=1}^m a_i \varphi_s(f^{m-i}(x)) - F(x)\| \leq \delta, \quad \forall x \in S,$$

then equation

$$\varphi(f^m(x)) = a_1 \varphi(f^{m-1}(x)) + \dots + a_{m-1} \varphi(f(x)) + a_m \varphi(x) + F(x)$$

has a unique solution  $\varphi: S \rightarrow X$  such that

$$\|\varphi_s(x) - \varphi(x)\| \leq \frac{\delta}{\| |r_1| - 1 | \cdot \dots \cdot \| |r_m| - 1 |}, \quad \forall x \in S.$$

## Pál Burai

University of Debrecen, Debrecen, Hungary

### On the equivalence of equations involving means and the solution to a problem of Daróczy

In this work we prove the equivalence of the following two functional equations:

$$f(\mathcal{A}(x, y; p)) + f(\mathcal{H}(x, y; 1 - p)) = f(x) + f(y) \quad x, y \in I,$$

and

$$2f(\mathcal{G}(x, y)) = f(x) + f(y) \quad x, y \in I.$$

Here  $I$  is a nonempty open interval on the positive real line, and  $\mathcal{A}(x, y; p)$ ,  $\mathcal{H}(x, y; 1 - p)$ ,  $\mathcal{G}(x, y)$  the weighted arithmetic mean with the weight  $p$ , the weighted harmonic mean with the weight  $1 - p$ , and the geometric mean respectively.

#### References

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## Jacek Chmieliński

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### Orthogonality preserving property, Wigner equation and stability

We deal with the stability of the orthogonality preserving property in the class of mappings phase-equivalent to linear or conjugate-linear ones. We give a characterization of approximately orthogonality preserving mappings in this class and we show some connections between the considered stability and the stability of the Wigner equation.

## Jacek Chudziak

University of Rzeszów, Rzeszów, Poland

### Stability of the Gołab-Schinzel functional equation

Let  $X$  be a linear space over a field  $K$  of real or complex numbers. At the 38th International Symposium on Functional Equations (2000, Noszvaj, Hungary) Professor Roman Ger raised, among others, the problem of Hyers-Ulam stability of the Gołab-Schinzel functional equation

$$f(x + f(x)y) = f(x)f(y) \tag{1}$$

for  $x, y \in X$ . In [1] it has been proved that in the class of functions  $f: X \rightarrow K$  satisfying some weak regularity assumptions, the equation (1) is superstable, i.e., every solution of the inequality

$$|f(x + f(x)y) - f(x)f(y)| \leq \varepsilon \tag{2}$$

for  $x, y \in X$ , where  $\varepsilon$  is a fixed nonnegative real number, either is bounded or satisfies (1). However, it is known (cf. [2, 3]) that the phenomenon of superstability is caused by the fact that we mix two operations. Namely, on the right-hand side of (1) we have the product, but in (2) we measure the distance between the two sides of (1) using the difference. Therefore, it is more natural, to measure the difference between 1 and the quotients of the sides of the equation (1). In the present talk we deal with this problem.

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## Marek Czerni

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### Comparison theorems for nonlinear functional inequalities

We present some comparison theorems for the solutions  $\psi$  of nonlinear functional inequalities

$$\begin{cases} \psi[f(x)] \leq G(x, \psi(x)), \\ (-1)^p \psi[f^{2M}(x)] \leq (-1)^p g_{2M}(x, \psi(x)) \end{cases}$$

where  $p \in \{0, 1\}$ ,  $M$  is a fixed positive integer and functional sequence  $g_n$  is defined by the recurrent formula

$$\begin{cases} g_0(x, y) = y, \\ g_{n+1}(x, y) = G(f^n(x), g_n(x, y)), \quad n = 0, 1, 2, \dots \end{cases}$$

In the talk we shall consider the case when the given function  $G$  is strictly decreasing with respect to the second variable. Moreover, we shall obtain a characterization of this results in terms of lattice theory.

## Thomas M.K. Davison

McMaster University, Hamilton, Ontario, Canada

### On the functional equation $g(xy) + g(x\tau(y)) = 2g(x)g(y)$

A function  $g$  on  $G$  is basic if  $\{u \in G : g(xuy) = g(xy), \text{ for all } x, y \text{ in } G\} = \{e\}$ . Using Stetkær's results [1], we prove that if  $g$  is basic, satisfies our equation, and the domain  $G$  of  $g$  is non-abelian then the centre of  $G$  is  $\text{Fix}(\tau) := \{z \in G : \tau(z) = z\}$ .

#### Reference

- [1] H. Stetkær, *D'Alembert's functional equations on metabelian groups*, Aequationes Math. **59** (2000), 306-320.

## Joachim Domsta

Gdańsk University of Technology, Gdańsk–Wrzeszcz, Poland

### Regular iteration of homeomorphisms of intervals

According to the known results, the regular iteration of a self mapping  $f$  of  $\mathbb{R}_+ := (0, \infty)$  without fixed points is closely related to regularly varying solutions  $\Psi$  of the corresponding Schröder equation

$$g(\Psi(x)) = \Psi(f(x)) \quad \text{for } x \in \mathbb{R}_+,$$

where  $g$  is a multiplication by a positive constant, not equal to 1. In more general cases,  $g$  is also a selfmapping of  $\mathbb{R}_+$  and the equation states a (weak) conjugacy of  $g$  to  $f$ . For regular  $\Psi$  joining  $f$  and  $g$  the relation between the regular iteration groups of  $f$  and  $g$  is analysed.

## Lyudmila S. Efremova

University of Nizhny Novgorod, Nizhny Novgorod, Russia

### On homoclinic points of $C^1$ -smooth skew products of interval maps

The structure of a neighborhood of the transverse homoclinic trajectory to the saddle periodic orbit of a  $C^1$ -smooth skew product of interval maps is investigated.

In the set of  $C^1$ -smooth  $\Omega$ -stable skew products of interval maps the criterion of the existence of a homoclinic trajectory is proved.

In the space of  $C^1$ -smooth skew products of interval maps the subset is distinguished in which the maps with transverse homoclinic trajectories to saddle periodic orbits are everywhere dense.

The examples of  $C^1$ -smooth skew products belonging to the boundary of the  $\Omega$ -stability domain and having the “exotic” properties are given.

The author is partially supported by RFBR, grant No 04-01-00457.

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## Danièle Fournier-Prunaret

LESIA-INSA, Toulouse, France

### Attractors bifurcations and basins in two-dimensional and three-dimensional biological models based on logistic maps

Joint work with Ricardo Lopez-Ruiz.

We consider 2-D and 3-D biological models given by coupling between logistic maps. The considered maps are noninvertible. We study the evolution of attractors (periodic orbits and chaotic attractors) and their basins when parameters change. An important tool is that of critical manifolds, which are specific to noninvertible maps and separate the state space in areas where points have a different number of preimages.

## Roman Ger

Silesian University, Katowice, Poland

### On a functional congruence related to Gołąb-Schinzel equation

Anna Mureńko, in her doctoral dissertation [\*] devoted to the functional equation

$$f(x + M(f(x))y) = f(x) \circ f(y),$$

has come across a functional congruence

$$F(x \circ y) - M(x)F(y) - F(x) \in T.$$

We are looking for a readable description of the solutions of that congruence in the case where given a linear space  $X$  over a field  $K$ , a subgroup  $(T, +)$  of the additive group  $(X, +)$  and a groupoid  $W \subset K \setminus \{0\}$ ,  $F$  maps  $W$  into  $X$  and  $M$  stands for a selfmapping of  $K$ .

## Reference

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## Attila Gilányi

University of Debrecen, Debrecen, Hungary

### Bernstein–Doetsch and Sierpiński theorems for $(M, N)$ -convex functions

Joint work with Zsolt Páles.

One of the classical results of the theory of convex functions is the theorem of F. Bernstein and G. Doetsch [1] which states that if a real valued Jensen-convex function defined on an open interval  $I$  is locally bounded above at one point in  $I$  then it is continuous. According to a related result by W. Sierpiński [4], the Lebesgue measurability of a Jensen-convex function implies its continuity, too.

In this talk we generalize the theorems above for  $(M, N)$ -convex functions, calling a function  $f: I \rightarrow J$ ,  $(M, N)$ -convex (c.f., e.g.: [3]) if it satisfies the inequality

$$f(M(x, y)) \leq N(f(x), f(y))$$

for all  $x, y \in I$ , where  $I$  and  $J$  are open intervals,  $M$  and  $N$  are suitable means on  $I$  and  $J$ , respectively. Our statements also generalize T. Zgraja's recent results on  $(M, M)$ -convex functions (c.f.: [5]). A special case of our theorems was presented in [2].

## References

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## Alina Gleska

Poznań University of Technology, Poznań, Poland

### Oscillatory properties of solutions of nonhomogeneous difference equations

Joint work with Jarosław Werbowski.

Our goal in this paper is to investigate the monotonic and oscillatory properties of solutions of the nonhomogeneous difference equation

$$(-1)^z \Delta^m y(n) = f(n, y(r_1(n)), y(r_2(n)), \dots, y(r_k(n))), \quad (\mathbf{E}_z)$$

where  $z, k \in \mathbb{N}$ ,  $m \geq 2$ ,  $r_i: \mathbb{N}_{n_0} \rightarrow \mathbb{N}_{n_0}$ ,  $\lim_{n \rightarrow \infty} r_i(n) = \infty$  and the function  $f: \mathbb{N}_{n_0} \times \mathbb{R}^k \rightarrow \mathbb{R}$  satisfies the condition

$$f(n, x_1, x_2, \dots, x_k) \operatorname{sgn} x_1 \geq \sum_{i=1}^k p_i(n) |x_i|, \quad (\text{C})$$

where  $p_i: \mathbb{N}_{n_0} \rightarrow \mathbb{R}_+ \cup \{0\}$  ( $i = 1, 2, \dots, k$ ).

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## Dorota Głazowska

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### An invariance of geometric mean with respect to Lagrangean means

Joint work with Janusz Matkowski.

The invariance of the geometric mean  $G$  with respect to the Lagrangean mean-type mapping  $(L^f, L^g)$ , i.e., the equation  $G \circ (L^f, L^g) = G$ , is considered. We show that the functions  $f$  and  $g$  must be of high class regularity. This fact allows to reduce the problem to a differential equation and determine the second derivatives of the generators  $f$  and  $g$ .

## Dijana Ilišević

University of Zagreb, Zagreb, Croatia

### Quadratic functionals and sesquilinear forms

The problem of the representability of quadratic functionals by sesquilinear forms arises from the well-known Jordan – von Neumann characterization of inner product spaces among normed spaces via the parallelogram identity. The aim of this talk is to present some recent algebraic Jordan – von Neumann type theorems in the setting of modules over involutive rings and algebras.

## Eliza Jabłońska

University of Technology, Rzeszów, Poland

### Solutions of a generalized Gołąb-Schinzel functional equation

Let  $X$  be a linear space over  $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ . We consider solutions  $f: X \rightarrow \mathbb{K}$  and  $M: \mathbb{K} \rightarrow \mathbb{K}$  of the functional equation

$$f(x + M(f(x))y) = f(x)f(y) \quad \text{for } x, y \in X \quad (1)$$

such that  $f$  is bounded on a set “big” in some sense. As a consequence we obtain measurable in Lebesgue and Baire sense solutions of (1). Our results refer to results of C.G. Popa and J. Brzdęk.

## Justyna Jarczyk

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### Invariance in the class of quasi-arithmetic means with function weights

Let  $I \subset \mathbb{R}$  be an open interval. Given a function  $\mu: I \times I \rightarrow (0, 1)$  and a strictly monotonic function  $\varphi: I \rightarrow \mathbb{R}$  we consider the mean  $M_\mu^\varphi: I \times I \rightarrow \mathbb{R}$  defined by

$$M_\mu^\varphi(x, y) = \varphi^{-1}(\mu(x, y)\varphi(x) + (1 - \mu(x, y))\varphi(y)).$$

We study the invariance of  $M_\lambda^{\text{id}}$  in such a class of means, that is the functional equation

$$\lambda(x, y)M_\mu^\varphi(x, y) + (1 - \lambda(x, y))M_\nu^\psi(x, y) = \lambda(x, y)x + (1 - \lambda(x, y))y.$$

In particular, we are interested in the case when  $\lambda(x, y) = \frac{r(x)}{r(x)+r(y)}$ ,  $\mu(x, y) = \frac{s(x)}{s(x)+s(y)}$  and  $\nu(x, y) = \frac{t(x)}{t(x)+t(y)}$  for every  $x, y \in I$ , where  $r, s, t$  are given positive functions on  $I$ . As a special case we obtain a recent result of J. Domsta and J. Matkowski [Aequationes Math. **71** (2006), 70–85; Theorem 2].

In particular, we come also to the Bajraktarević means. They satisfy

$$M_\mu^\varphi(x, y) + M_{1-\mu}^\psi(x, y) = x + y, \quad x, y \in I,$$

i.e., the arithmetic mean is invariant with respect to  $(M_\mu^\varphi, M_{1-\mu}^\psi)$ .

## Witold Jarczyk

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### Almost convex functions on Abelian groups

Joint work with Miklós Laczkovich.

A  $\sigma$ -ideal  $\mathcal{I}$  in  $\mathbb{R}^n$  is called linearly invariant if

$$A \in \mathcal{I} \text{ and } x \in \mathbb{R}^n \text{ imply } x - A \in \mathcal{I}.$$

We say that  $\sigma$ -ideals  $\mathcal{I}_1$  and  $\mathcal{I}_2$  in  $\mathbb{R}^n$  and  $\mathbb{R}^n \times \mathbb{R}^n$ , respectively, are conjugate if they fulfil the following Fubini condition:

for every  $A \in \mathcal{I}_2$  the sections  $\{y \in \mathbb{R}^n : (x, y) \in A\}$  are in  $\mathcal{I}_1$  for  $\mathcal{I}_1$ -a.a.  $x \in \mathbb{R}^n$ .

In 1970 Marek Kuczma published the following result [Colloq. Math. **21** (1970), 279-284]. Here  $\phi$  stands for the transformation of  $\mathbb{R}^n \times \mathbb{R}^n$  defined by  $\phi(x, y) = (x + y, x - y)$ .

*Let  $\mathcal{I}_1$  and  $\mathcal{I}_2$  be conjugate proper linearly invariant  $\sigma$ -ideals in  $\mathbb{R}^n$  and  $\mathbb{R}^{2n}$ , respectively, fulfilling the conditions*

if  $A \in \mathcal{I}_1$  then  $aA \in \mathcal{I}_1$  for every  $a \in \mathbb{R}$ ,

if  $A \in \mathcal{I}_2$  then  $\phi^{-1}(A) \in \mathcal{I}_2$ .

If  $D \subset \mathbb{R}^n$  is an open convex set and  $f: D \rightarrow \mathbb{R}$  is an  $\mathcal{I}_2$ -almost convex function, i.e.,

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$$

for  $\mathcal{I}_2$ -a.a.  $(x, y) \in D$ , then there exists a unique Jensen convex function  $g: D \rightarrow \mathbb{R}$  such that  $g(x) = f(x)$  for  $\mathcal{I}_1$ -a.a.  $x \in D$ .

We present a generalization of Kuczma's result for functions defined on a subset of an Abelian group  $G$ . Convexity [almost convexity] of  $f: A \rightarrow \mathbb{R}$  means here that the inequality

$$2f(x) \leq f(x+h) + f(x-h)$$

holds for all [ $\mathcal{I}_2$ -a.a.]  $(x, y) \in G \times G$  such that  $x, x+h, x-h \in A$ .

## Hans-Heinrich Kairies

Clausthal University of Technology, Clausthal-Zellerfeld, Germany

### On some problems concerning a sum type operator

The sum type operator  $F$ , given by

$$F[\varphi](x) := \sum_{k=0}^{\infty} 2^{-k} \varphi(2^k x),$$

has been thoroughly discussed in the last years. Nevertheless, there remained some open problems. We state some of them which are connected with

1. Images and pre- images of  $F$ ,
2. Spectral properties of  $F$ ,
3. The maximal domain of  $F$ ,
4. Characterizations of  $F[\varphi]$ ,
5. A two parameter extension of  $F$ .

## Zoltán Kaiser

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### On stability of the Fréchet equation

Let  $X$  be a linear space and  $Y$  be a normed space over the field of rational numbers. The stability problem concerning the Fréchet equation is the following:

Let the  $n$ -th differences of the function  $f: X \rightarrow Y$  are bounded, i.e.,

$$\|\Delta_{y_1, \dots, y_n} f(x)\| \leq \varepsilon \quad (y_1, \dots, y_n, x \in X) \quad (1)$$

for some  $\varepsilon > 0$ . Is there any generalized polynomial  $g$  of degree at most  $n - 1$ , for which  $f - g$  is bounded?

Without any regularity condition of  $f$ , the first positive answer of this problem was given by D.H. Hyers [3]. M. Albert and J.A. Baker [1] gave a result in a more general form, with a shorter proof.

C. Borelli and C. Invernizzi [2] dealt with the stability of the Fréchet equation in the case that the right hand-side of (1) is an  $\alpha$ -homogeneous function, but there was a mistake in the proof of the main theorem. Motivated by these results, we prove a stability theorem of the Fréchet equation in Banach spaces over fields with valuation.

## References

- [1] M. Albert, J.A. Baker, *Functions with bounded  $n$ th differences*, Ann. Polon. Math. **43** (1983), 93–103.
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- [3] D.H. Hyers, *Transformations with bounded  $m$ th differences*, Pacific J. Math. **11** (1961), 591–602.

**Barbara Kocłęga-Kulpa**

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## On some equation connected with Hadamard inequalities

Joint work with Tomasz Szostok.

We consider some equations connected with Hadamard inequalities. Namely, we observe that the function  $f(x) = x^2$  satisfies the condition

$$\int_x^y f(t) dt = (y - x) \left[ \frac{2}{3} f\left(\frac{x + y}{2}\right) + \frac{1}{6} f(x) + \frac{1}{6} f(y) \right]. \quad (1)$$

We ask about functions having properties of this kind. Moreover, we present some generalization of the equation (1), i.e.,

$$f(y) - g(y) = (y - x)[h(x + y) + \phi(x) + \psi(y)],$$

which was considered in [1] for functions acting on  $\mathbb{R}$ . We determine all solutions of this equation in more general case — for integral domains.

## Reference

- [1] T. Riedel, P.K. Sahoo, *Mean value theorems and functional equations*, World Scientific, Singapore - New Jersey - London - Hong Kong, 1998.

## Imre Kocsis

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### A bisymmetry equation on restricted domain

Let  $X \subset \mathbb{R}$  be an interval of positive length and define the set  $\Delta = \{(x, y) \in X \times X \mid x \geq y\}$ . In this note we give the solution of the equation

$$F(G_1(x, y), G_2(u, v)) = G(F(x, u), F(y, v)), \quad (x, y) \in \Delta, (u, v) \in \Delta,$$

where the functions  $F: X \times X \rightarrow X$ ,  $G_1: \Delta \rightarrow X$ ,  $G_2: \Delta \rightarrow X$ , and  $G: F(X, X) \times F(X, X) \rightarrow X$  are continuous and strictly monotonic (in the same sense) in each variable. The result is a generalization of a previous one investigated by the author (under publication in *Aequationes Mathematicae*). The original problem was published by R. Duncan Luce and J.A. Marley in *The Journal of Risk and Uncertainty* (**30:1** (2005), 21–62).

## Dorota Krassowska

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### On a nonlinear simultaneous system of functional inequalities

Under some conditions on given real functions  $f, F, g, G$  we determine all the continuous at least at one point solutions  $\varphi$  of the simultaneous system of functional inequalities

$$\begin{cases} \varphi(f(x)) \leq F(\varphi(x)) \\ \varphi(g(x)) \leq G(\varphi(x)) \end{cases}, \quad x \in I,$$

where  $I \subset \mathbb{R}$  is an arbitrary interval.

## Xiaopei Li

Zhanjiang Normal University, Zhanjiang, China

### An iterative equation on the unit circle

Joint work with Shengfu Deng.

A functional equation of nonlinear iterates is discussed on the circle  $S^1$  for its continuous solutions and differentiable solutions. By lifting to  $\mathbb{R}$ , the existence, uniqueness and stability of those solutions are obtained. Techniques of continuation are used to guarantee the preservation of continuity and differentiability in lifting.

## Arkadiusz Lisak

University of Information Technology and Management, Rzeszów, Poland

### A characterization of some operators by functional equations

Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  and let  $r_n$  and  $q_n$  be two sequences of real numbers for every  $n \in \mathbb{N}$ . We define for fixed  $t \neq 0$ ,  $y, z \in \mathbb{R}$ ,  $y \neq z$  sequence of operators  ${}_t\phi_{y,z}^{(n)}$  in the following way

$${}_t\phi_{y,z}^{(1)}f(x) = t \frac{f\left(\frac{x+y}{t}\right) - f\left(\frac{x+z}{t}\right)}{y-z},$$

$${}_t\phi_{y,z}^{(n+1)}f = t[r_n {}_t\phi_{y,z}^{(n)}f + q_n {}_t\phi_{2y,2z}^{(n)}f]$$

for  $x \in \mathbb{R}$ . For every  $n \in \mathbb{N}$  we consider functional equations

$${}_t\phi_{y,z}^{(n)}f(x) = g(x),$$

where  $f, g: \mathbb{R} \rightarrow \mathbb{R}$  are unknown functions, and we solve them in special cases. One of the special cases has been dealt with in [1].

Next we consider and solve the equations on the Abelian groups. We show that the equations characterize polynomials (or generalized polynomials) and their derivatives (or homomorphisms of a special form).

#### Reference

- [1] T. Riedel, M. Sablik, A. Sklar, *Polynomials and divided differences*, Publ. Math. Debrecen **66** (2005), 313–326.

## László Losonczy

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### Polynomials with all zeros on the unit circle

Joint work with P. Lakatos.

We summarize recent results on polynomials all of whose zeros are on the unit circle. We give sufficient conditions for this and also necessary conditions (in terms of the coefficients), describe the methods used. Finally we mention a very general new sufficient condition for self-inversive polynomials.

## Grażyna Łydzzińska

Silesian University, Katowice, Poland

### On some set-valued iteration semigroups

Let  $X$  be an arbitrary set. We present the necessary and sufficient conditions for a set-valued function  $A: X \rightarrow 2^{\mathbb{R}}$  under which a family of multifunctions of the form

$$A^{-1}(A(x) + \min\{t, q - \inf A(x)\})$$

where  $q := \sup A(X)$ , naturally occurring in the iteration theory, is an iteration semigroup.

## Andrzej Mach

Świętokrzyska Academy, Kielce, Poland

### On some functional equations involving with involution

Joint work with Zenon Moszner.

We present some theorems characterizing solutions of the equation  $f(x) = f(\varphi(x)) + g(x)$ , where  $\varphi$  is a given involution, and particularly differentiable solutions of the equation  $f(x) = f(1-x) + 2x - 1$ . The stability of this equation and non-existence of the extremal points of the set of solutions are proved.

#### References

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- [3] P. Volkmann, *Caractérisation de la fonction  $f(x) = x$  par un système de deux équations fonctionnelles*, C. R. Math. Rep. Acad. Sci. Canada, **5** (1983), 27–28.

## Elena Makhrova

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### Dendrites with the periodic points property

Dendrite is a locally connected continuum without subsets homeomorphic to a circle.

A dendrite  $X$  is said to have the periodic points property provided that for any continuous map  $f: X \rightarrow X$  and for an arbitrary subcontinuum  $Y \subset X$  such that  $Y \subseteq f(Y)$  the last inclusion implies  $Y \cap \text{Per}(f) \neq \emptyset$ ,  $\text{Per}(f)$  is the periodic points set of  $f$ .

In [1] it is shown that a finite tree (a dendrite with a finite ramification points set and points of finite order) has the periodic points property, the example of a dendrite which has no the periodic points property is constructed.

In the report the structure of dendrites having the periodic points property is investigated. The next result is true.

#### THEOREM A

*Let  $X$  be a dendrite such that the derivative of the ramification points set of  $X$  is at most countable. Then  $X$  has the periodic points property.*

Theorem A does not occur if the derivative of the ramification points set of a dendrite  $X$  is uncountable. The example of such dendrite is constructed here.

Necessary conditions of structure of dendrites having the periodic points property are presented.

This research is partially supported by RFBR, grant No 04-01-00457.

## Reference

- [1] E.N. Makhrova, *On the existence of periodic points of continuous maps of dendrites*, Some Problems of Fundamental and Applied Mathematics, Moscow 2006 (to appear).

### Janusz Matkowski

Silesian University, Katowice, Poland

#### On extension of solutions of simultaneous systems of functional equations

Some sufficient conditions which allow to extend every local solution of a simultaneous system of equations to a global one are presented.

### Fruzsina Mészáros

University of Debrecen, Debrecen, Hungary

#### Functional equations on group

Joint work with Zs. Ádám, K. Lajkó and Gy. Maksa.

Let  $G$  be an arbitrary group written additively. We give the general solution of the functional equation

$$f(x)f(x+y) = f(y)^2 f(x-y)^2 g(y) \quad (x, y \in G)$$

and all the solutions of

$$f(x)f(x+y) = f(y)^2 f(x-y)^2 g(x) \quad (x, y \in G)$$

with the additional supposition  $g(x) \neq 0$  for all  $x \in G$ . In both cases  $f, g: G \rightarrow \mathbb{R}$  are unknown functions.

### Janusz Morawiec

Silesian University, Katowice, Poland

#### On a refinement type equation

Joint work with Rafał Kapica.

Let  $(\Omega, \mathcal{A}, P)$  be a probability space. We show that the trivial function is the unique  $L^1$ -solution of the following refinement type equation

$$f(x) = \int_{\Omega} \left| \frac{\partial A}{\partial x}(x, \omega) \right| f(A(x, \omega)) dP(\omega)$$

in a wide class of the given functions  $A$ . This class contains functions of the form  $A(x, \omega) = \alpha(\omega)x - \beta(\omega)$  with  $-\infty < \int_{\Omega} \log |\alpha(\omega)| dP(\omega) < 0$ .

## Jacek Mrowiec

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### Generalized convex functions in linear spaces

The notion of generalized convex functions has been introduced by E.F. Beckenbach in the following way:

Let  $\mathcal{F}$  be a two-parameter family of continuous real-valued functions defined on an open interval  $(a, b)$  such that for any two distinct points  $x_1, x_2 \in (a, b)$  and any  $t_1, t_2 \in \mathbb{R}$  there exists exactly one  $\varphi = \varphi_{(x_1, t_1)(x_2, t_2)} \in \mathcal{F}$  satisfying

$$\varphi(x_i) = t_i, \quad i = 1, 2.$$

We say that a function  $f: (a, b) \rightarrow \mathbb{R}$  is  $\mathcal{F}$ -convex if for any distinct  $x_1, x_2 \in (a, b)$

$$f(x) \leq \varphi_{(x_1, f(x_1))(x_2, f(x_2))}(x) \quad \text{for every } x \in [x_1, x_2].$$

We present a method for generating two-parameter families not only on the real line. This method allows us to extend the notion of generalized convex functions to linear spaces. Such functions have many properties of functions, which are convex in the usual sense, and proofs of their properties are easy.

## Anna Mureńko

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### On solutions of some conditional generalizations of the Gołąb-Schinzel equation

We deal with the conditional functional equations

$$\text{if } x, y, x + M(g(x))y > 0, \text{ then } g(x + M(g(x))y) = g(x) \circ g(y),$$

$$\text{if } x, y, x + M(g(x))y > 0, \text{ then } g(x + M(g(x))y) = g(x)g(y),$$

where  $M: \mathbb{R} \rightarrow \mathbb{R}$ ,  $\circ: \mathbb{R}^2 \rightarrow \mathbb{R}$  and  $g: (0, \infty) \rightarrow \mathbb{R}$  is Lebesgue measurable or Baire measurable. We consider the above equations under some additional (different for each equation) assumptions.

## Veerapazham Murugan

Indian Institute of Technology, Madras, India

### Smooth solutions for a functional equation involving series of iterates

Joint work with P.V. Subrahmanyam.

In this paper we give sufficient conditions for the existence and uniqueness of  $C^2$  solution for the functional equation

$$\sum_{i=1}^{\infty} \lambda_i f^i(x) = F(x), \quad x \in [a, b] \subset \mathbb{R},$$

where  $\lambda_i$ 's are nonnegative real numbers with  $\sum_{i=1}^{\infty} \lambda_i = 1$  and  $F$  is a given  $C^2$  function on  $[a, b]$  satisfying some additional conditions. Such functional equations have been studied earlier by Kulczycki, Shengfu, Tabor, Xiaopei, Żołądek and the present authors.

## Adam Najdecki

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### On the stability of some generalization of Cauchy's, d'Alembert's and quadratic functional equations

Let  $X \neq \emptyset$  be a set,  $(Y, +)$  be a commutative semigroup with a complete invariant metric,  $k \in \mathbb{N}$  and let  $A, B: Y \rightarrow Y$ ,  $G_i: X \times X \rightarrow X$  for  $i \in \{1, 2, \dots, k\}$ . We consider stability of the functional equation

$$\sum_{i=1}^k f(G_i(x, y)) = A(f(x)) + B(f(y))$$

in the class of functions  $f: X \rightarrow Y$ , as well as of the equation

$$\sum_{i=1}^k f(G_i(x, y)) = kf(x)f(y)$$

in the class of functions mapping  $X$  into a real or complex normed algebra with a multiplicative norm.

## Wiesława Nowakowska

Poznań University of Technology, Poznań, Poland

### Sufficient conditions for the oscillation of solutions of iterative functional equations

Joint work with Jarosław Werbowski.

Sufficient conditions for the oscillation of all solutions of iterative functional equations will be presented. Oscillation criteria for difference equations will be obtained.

## Andrzej Olbryś

Silesian University, Katowice, Poland

### A characterization of $(t_1, \dots, t_n)$ -Wright affine functions

Let  $t_1, \dots, t_n$ ,  $n \geq 2$ , be fixed positive numbers, let  $X$  be a linear space over the field  $L(t_1, \dots, t_n)$  generated by  $t_1, \dots, t_n$  (i.e., the smallest field containing the set  $\{t_1, \dots, t_n\}$ ) and let  $Y$  be a commutative group.

Following [1] where the definition of  $(t_1, \dots, t_n)$ -Wright convex function was given we introduce the definition of  $(t_1, \dots, t_n)$ -Wright affine function as a function  $f: D \rightarrow Y$  satisfying the following functional equation:

$$\Delta_{t_1 z, \dots, t_n z} f(x) = 0, \quad (x, z) \in D \times X : x + (t_1 z + \dots + t_n z) \in D,$$

where  $D \subset X$  is a  $L(t_1, \dots, t_n)$ -convex set.

In the paper [2] K. Lajkó has given a characterization of  $(t, 1-t)$ -Wright affine functions. We extend this result to  $(t_1, \dots, t_n)$ -Wright affine functions of an arbitrary order.

## References

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- [3] A. Olbryś, *A characterization of  $(t_1, \dots, t_n)$ -Wright affine functions*, submitted.

**Ágota Orosz**

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## Difference equations on discrete polynomial hypergroups

In the classical theory of difference equations the translate of a function by  $n$  and the translation of the function  $n$ -times by 1 give the same result for all  $n$  in  $\mathbb{N}$ . But in the hypergroup case there are two different ways to define difference equations along these two interpretations. In this talk we give the solutions of a homogenous linear difference equation of order  $N$  on a polynomial hypergroup (which is actually a difference equation with nonconstant coefficients in the classical sense) in both cases.

**Boris Paneah**

Technion, Haifa, Israel

## Strong stability of functional equations in several variables

We deal with compact supported Banach - valued functions  $F$  satisfying

$$(\mathcal{P}F)(x, y) := F(a(x, y)) - \sum_{j=1}^n \alpha_j(x, y) F(a_j(x, y)) = H(x, y)$$

for all  $(x, y)$  in a bounded domain  $D \subset \mathbb{R}^2$ . The prototype is the Jensen equation with  $a = \alpha_1 x + \alpha_2 y$ ,  $a_1 = x$ ,  $a_2 = y$ , and real positive numbers  $\alpha_1, \alpha_2$  satisfying  $\alpha_1 + \alpha_2 = 1$ . The problem of its stability goes back to Ulam (1940). Our approach is novel in two essential ways.

The first is that if  $a = \sum \alpha_j a_j$  on a one-dimensional submanifold  $\Gamma \subset \overline{D}$  (weak Jensen operator  $\mathcal{P}$ ), then under quite general conditions the stability problem for  $\mathcal{P}$  is overdetermined: the smallness of  $H$  only on  $\Gamma$  implies the nearness of  $F$  to a linear function (strong stability).

The second is a functional analytic point of view. We consider the linear operator  $\mathcal{P}_\Gamma$  – the restriction of  $\mathcal{P}$  to  $\Gamma$  – between appropriate function spaces and give conditions of its surjectivity. The stability then follows from functional analytic considerations.

**Iwona Pawlikowska**

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## Flett-type means II

We continue the investigation of properties of means obtained from Flett's mean value theorem. We take into account two generalizations of Flett mean value theorem and we show that means obtained from these theorems are unique. There is also discussed problem of equivalence Flett-type means to the well known means.

**Zsolt Páles**

University of Debrecen, Hungary

## A regularity problem concerning the equality of generalized quasi-arithmetic means

Let  $I \subset \mathbb{R}$  be a nonvoid open interval. Given a continuous strictly monotone function  $\varphi: I \rightarrow \mathbb{R}$  and a Borel probability measure  $\mu$  on  $[0, 1]$ , the mean  $M_{\varphi, \mu}: I^2 \rightarrow I$  is defined by

$$M_{\varphi, \mu}(x, y) := \varphi^{-1} \left( \int_0^1 \varphi(tx + (1-t)y) d\mu(t) \right) \quad (x, y \in I).$$

The equality problem of these means is to describe all pairs  $(\varphi, \mu)$  and  $(\psi, \nu)$  such that

$$M_{\varphi, \mu}(x, y) = M_{\psi, \nu}(x, y) \quad (x, y \in I).$$

By a recent result obtained jointly with Z. Makó, if there exists a point  $p \in I$  such that  $\varphi$  and  $\psi$  are differentiable at  $p$  and  $\varphi'(p)\psi'(p) \neq 0$ , then a necessary condition for the above equality problem is that the first moments of the measures  $\mu$  and  $\nu$  be equal, i.e.,

$$\mu_1 := \int_0^1 t d\mu(t) = \int_0^1 t d\nu(t) =: \nu_1.$$

Introducing the notion of quasi-differentiability, we deduce this (and more general conditions) under much weaker assumptions.

**Magdalena Piszczek**  
Pedagogical University, Kraków, Poland

## On a multivalued second order differential problem

Let  $K$  be a closed convex cone with the nonempty interior in a real Banach space and let  $cc(K)$  denote the family of all nonempty convex compact subsets of  $K$ . Assume that continuous linear multifunctions  $H, \Psi: K \rightarrow cc(K)$  are given. We consider the following problem

$$\begin{aligned} D^2 \Phi(t, x) &= \Phi(t, H(x)), \\ D \Phi(t, x)|_{t=0} &= \{0\}, \\ \Phi(0, x) &= \Psi(x) \end{aligned}$$

for  $t \geq 0$  and  $x \in K$ , where  $D \Phi(t, x)$  denotes the Hukuhara derivative of  $\Phi(t, x)$  with respect to  $t$  and  $D^2 \Phi(t, x) = D(D \Phi(t, x))$ .

**Barbara Przebieracz**  
Silesian University, Katowice, Poland

## Near iterability

Inspired by Problem (3.1.12) posed by E. Jen in [T] we present various approaches to the concept of near-iterability. We deal with selfmappings of a real compact interval, characterize and compare a few classes of near-iterable functions in a sense. That includes

- almost iterable functions, that is continuous  $f: X \rightarrow X$ , for which there exists an iterable  $g: X \rightarrow X$  such that

$$f^n - g^n \text{ converges to } 0 \text{ everywhere in } X \tag{1}$$

and the convergence is uniform on every interval with endpoints being two consecutive fixed points of  $f$  (cf. [J]);

- functions satisfying (1);

and some weaker condition than (1), that is

- functions  $f$  for which there exists an iterable  $g$  such that

$$f^n(x) - g^n(x) \text{ converges to } 0 \text{ for every } x \in X \setminus M, \tag{2}$$

where the set  $M$  has empty interior;

- approximately iterable functions, that is continuous  $f: X \rightarrow X$ , such that for every  $\varepsilon > 0$  there exists an iterable function  $g: X \rightarrow X$  and a positive integer  $n_0$  satisfying the inequality  $|f^n(x) - g^n(x)| < \varepsilon$ ,  $n \geq n_0, x \in X$ ;
- closure of the set of all iterable functions.

(This is the continuation of my talk at 10th ICFEI).

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## Maciej Sablik

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### A generalization of generalized bisymmetry

Many authors (cf. e.g. the references below) have been concerned with functional equations of generalized bisymmetry. In particular, under some assumptions the form of solutions has been determined in the case where unknown functions map finite products of intervals into reals. In the present talk we determine (again, under some regularity assumptions) the form of operators  $M$  defined in some function spaces of measurable functions and satisfying a “Fubini type” equality

$$M_{[s]}M_{[t]}x(s, t) = M_{[t]}M_{[s]}x(s, t)$$

for every real function  $x$  such that  $x(s, \cdot)$ ,  $x(\cdot, t)$  belong to a given function space. The proofs use the results obtained earlier for the generalized bisymmetry.

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## Vsevolod Sakbaev

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### On the spaces of functions integrable with respect to finite additive measure and the generalized convergence

The investigation of qualitative properties of the dynamical systems and the ill-posed boundary-value problems by regularization methods lead to considering of divergent sequences. To describe the behavior of the divergent sequence by the regular methods of generalized summation and to obtain the distributions on the set of its limit points we consider the measures on the set of natural numbers  $\mathbb{N}$  which concentrated on the infinity. Then

any nonnegative normalized measure which concentrated on the infinity defines the regular method of generalized summation of sequences such that any bounded sequence is summable. We investigate the procedure of weak integration of vector-valued functions on the set with bounded additive measure and the properties of the spaces of integrable functions. The Hilbert space of square integrable function is applied to description of the regularization of ill-posed Cauchy problem ([1]).

This work is partially supported by RFBR, grant No 04-01-00457.

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## Adolf Schleiermacher

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### On real places

Let  $K$  be a formally real field and  $\varphi: K \rightarrow \mathbb{R} \cup \{\infty\}$  a mapping which satisfies

$$\varphi(a + b) = \varphi(a) + \varphi(b) \quad \text{and} \quad \varphi(a \cdot b) = \varphi(a) \cdot \varphi(b) \quad (*)$$

whenever the righthand sides in these equations are defined. Here the operations are extended as usual to the symbol  $\infty$  so that  $x + \infty = \infty$ ,  $y \cdot \infty = \infty$  for  $x, y \in \mathbb{R}$ ,  $y \neq 0$  and  $\infty \cdot \infty = \infty$  while  $\infty + \infty$  and  $0 \cdot \infty$  remain undefined. A mapping  $\varphi$  satisfying (\*) is called a real place if in addition  $\varphi(1) = 1$ . This last requirement serves to exclude trivial cases for which  $\varphi(K) \subseteq \{0, \infty\}$ . In  $K$  we introduce a partial ordering  $\leq_P$  defined as usual by its set  $P$  of non-negative elements. A real place  $\varphi$  will be called order preserving if for  $a \leq_P b$  and  $\varphi(a), \varphi(b) \in \mathbb{R}$  we have  $\varphi(a) \leq \varphi(b)$ . If  $\varphi(K) \subseteq \mathbb{R}$  then a real place  $\varphi$  is simply an isomorphic embedding of  $K$  in  $\mathbb{R}$ . The object of this talk is to investigate the fields  $K$  with partial ordering  $\leq_P$  for which all order preserving real places are embeddings.

It is known that this occurs when all orderings of  $K$  compatible with  $\leq_P$  are Archimedean. Other necessary and sufficient conditions will be obtained by studying various properties of the partial ordering  $\leq_P$ . For instance, we consider the topology  $\mathcal{T}$  defined by  $\leq_P$  or we study the unit interval  $T = \{x : -1 \leq_P x \leq_P 1\}$  as a convex set in the vector space  $K_{\mathbb{Q}}$  over  $\mathbb{Q}$ .

For a subset  $S$  of  $K_{\mathbb{Q}}$  the core of  $S$  consists of all  $x \in S$  such that each line through  $x$  contains an open segment  $(a, b)$  with  $x \in (a, b)$  and  $(a, b) \subseteq S$ . Here of course,  $(a, b) = \{\rho a + (1 - \rho)b : \rho \in \mathbb{Q} \text{ and } 0 < \rho < 1\}$ . Note that in infinite dimensional vector spaces there exist convex sets whose affine hull is the whole space but whose core is nevertheless empty. Conditions characterizing the fields for which all order preserving real places are embeddings are for instance:

1. The sequence  $\frac{1}{2^n}$  converges to zero with respect to topology  $\mathcal{T}$ .
2. In the vector space  $K_{\mathbb{Q}}$  the (convex) set  $T$  has non-empty core.
3. The topology  $\mathcal{T}$  can be defined by a spectral norm  $\mu$  satisfying  $\mu(1) = 1$ ,  $\mu(\lambda x) = |\lambda|\mu(x)$  for all  $x \in K$ ,  $\lambda \in \mathbb{Q}$ , and  $\mu(x) \leq 1$  if and only if  $x \in T$ .

## Stanisław Siudut

Pedagogical University, Kraków, Poland

### Stability of the Cauchy equation for convolutions

Let  $S$  be a real or complex normed algebra with multiplication  $*$  and let  $F$  be a  $S$ -valued function defined on  $S$ .

The equation

$$F(x * y) - F(x) * F(y) = 0 \quad (x, y \in S)$$

will be called superstable if for each  $F$  satisfying

$$\|F(x * y) - F(x) * F(y)\| \leq \delta, \quad x, y \in S,$$

where  $\delta > 0$ , either  $F$  is a bounded function or  $F(x * y) = F(x) * F(y)$  for all  $x, y \in S$ .

The above equation is not superstable for some functions algebras with convolution multiplication  $*$ . However, under some additional assumptions on the range  $F(S)$  of  $F$ , if the set  $\{F(x * y) - F(x) * F(y) : x, y \in S\}$  is bounded in  $S$  then  $F$  is a bounded function or  $\|F(x * y) - F(x) * F(y)\| = 0$  for all  $x, y \in S$ .

## Dariusz Sokołowski

Silesian University, Katowice, Poland

### Solutions with exponential character to a linear functional equation and roots of its characteristic equation

We deal with the functional equation

$$\varphi(x) = \int_S \varphi(x + M(s)) \sigma(ds) \quad (1)$$

and its characteristic equation

$$\int_S e^{\lambda M(s)} \sigma(ds) = 1 \quad (2)$$

assuming that  $(S, \Sigma, \sigma)$  is a finite measure space,  $M: S \rightarrow \mathbb{R}$  is a  $\Sigma$ -measurable bounded function and  $\sigma(M \neq 0) > 0$ . By a *solution* of (1) we mean a real function  $\varphi$  defined on a set of the form  $(a, +\infty) \cap W$  with  $W + \langle M(S) \rangle \subset W$ , and such that for every

$$x \in (a + \sup\{|M(s)| : s \in S\}, +\infty) \cap W$$

the integral  $\int_S \varphi(x + M(s)) \sigma(ds)$  exists and (1) holds. Our main result reads.

**THEOREM.**

Assume  $\lambda$  is a real number. If for some solution  $\varphi$  of (1) either a finite and nonzero limit

$$\lim_{x \rightarrow +\infty} \frac{\varphi(x)}{xe^{\lambda x}}$$

exists, or

$$0 < \liminf_{x \rightarrow +\infty} \frac{\varphi(x)}{e^{\lambda x}} \leq \limsup_{x \rightarrow +\infty} \frac{\varphi(x)}{e^{\lambda x}} < +\infty,$$

then  $\lambda$  is a root of (2).

## Paweł Solarz

Pedagogical University, Kraków, Poland

### Iterative roots for homeomorphisms with a rational rotation number

Let  $F: S^1 \rightarrow S^1$  be an orientation-preserving homeomorphism such that its rotation number is rational, i.e.,  $\alpha(F) = \frac{q}{n}$ , where  $q, n \in \mathbb{N}$ ,  $\gcd(q, n) = 1$  and  $q < n$ . Denote by  $\text{Per } F$  the set of all periodic points of  $F$  and let

$$\mathcal{M}_F^+ := \left\{ u \in \text{Per } F : \exists w \in \text{Per } F \setminus \{u\} : \left( \overline{(u, w)} \cap \text{Per } F = \emptyset \wedge \forall z \in \overline{(u, w)} \ z \in \overline{(u, F^n(z))} \right) \right\}.$$

The continuous and orientation-preserving solutions of the equation

$$G^m(z) = F(z), \quad z \in S^1,$$

where  $m \geq 2$  is an integer, exist if and only if some orientation-preserving continuous solution  $\Phi: \text{Per } F \rightarrow S^1$  of the equation

$$\Phi(F(z)) = e^{2\pi i \frac{q}{n}} \Phi(z), \quad z \in \text{Per } F$$

satisfies

$$e^{2\pi i \frac{q+jn}{mn}} \Phi[\text{Per } F] = \Phi[\text{Per } F] \quad \text{and} \quad e^{2\pi i \frac{q+jn}{mn}} \Phi[\mathcal{M}_F^+] = \Phi[\mathcal{M}_F^+]$$

for some  $j \in \{0, \dots, m-1\}$ .

## Tomasz Szostok

Silesian University, Katowice, Poland

### On some conditional Cauchy equation

Let  $X$  be a real inner product space. The Cauchy equation with the right-hand side multiplied by some constant is considered. This equation is assumed for all  $x, y \in X$  satisfying the equality  $\frac{\|x-y\|}{\|x+y\|} = \alpha$  where  $\alpha \in (0, \infty)$  is given. Solutions of this conditional equation under some assumptions are determined.

## Jacek Tabor

Jagiellonian University, Kraków, Poland

### Shadowing with multidimensional time in Banach spaces

As it is well-known, many stability problems can be reduced to the following one concerning difference equations:

PROBLEM.

Let  $(A_k)_{k \in \mathbb{Z}}$  be a sequence of bounded linear operators in a Banach space  $X$  and let  $(y_k)_{k \in \mathbb{Z}} \subset X$  be such that

$$\|y_{k+1} - A_k y_k\| \leq \delta \quad \text{for } k \in \mathbb{Z}.$$

Does there exist a solution to  $x_{k+1} = A_k x_k$  such that

$$\|x_k - y_k\| \leq \varepsilon?$$

The aim of the talk is to deal with the generalization of the above problem to the case of multidimensional time. Applying the Taylor functional calculus [3] we generalize the results from [2] to the case of Banach spaces [1].

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## Józef Tabor

University of Rzeszów, Rzeszów, Poland

### Restricted stability of the Cauchy equation in metric semigroups

Joint work with Jacek Tabor.

Let  $f: [0, \infty) \rightarrow [1, \infty)$  be defined by

$$f(x) := \frac{1}{3}x + 1.$$

Then

$$|f(x+y) - f(x) - f(y)| \leq 1 \quad \text{for } x, y \in [0, \infty),$$

but there is no additive function  $a: [0, \infty) \rightarrow [1, \infty)$  satisfying the condition

$$\sup_{x \in [0, \infty)} |f(x) - a(x)| < \infty.$$

However, function  $f$  can be approximated by an additive one on  $[3, \infty)$ . Namely, we have

$$\left| f(x) - \frac{1}{3}x \right| \leq 1 \quad \text{for } x \in [3, \infty).$$

Such an effect is caused by the fact that the division by 2 is not globally performable in the set  $[1, \infty)$ .

Our aim is to deal with the stability of the Cauchy equation in such a frame.

## Peter Volkmann

University of Karlsruhe, Karlsruhe, Germany

### Characterization of the absolute value of complex linear functionals by functional equations

Joint work with Karol Baron.

Let  $V$  be a complex vector space. The functions  $f(x) = |\varphi(x)|$  ( $x \in V$ ) with some linear  $\varphi: V \rightarrow \mathbb{C}$  are characterized by each of the two equations

$$\sup_{\alpha \in \mathbb{R}} f(x + e^{\alpha i}y) = f(x) + f(y) \quad \text{and} \quad \inf_{\alpha \in \mathbb{R}} f(x + e^{\alpha i}y) = |f(x) - f(y)|.$$

The paper will appear in Sem. LV (<http://www.mathematik.uni-karlsruhe.de/~semLv>).

## Janusz Walorski

Silesian University, Katowice, Poland

### On some solutions of the Schröder equation in Banach spaces

Let  $X$  be a Banach space. We consider the problem of existence and uniqueness of solutions of the Schröder equation

$$\varphi(f(x)) = A\varphi(x),$$

where the function  $f: X \rightarrow X$  and the bounded linear operator  $A: X \rightarrow X$  are given.

## Szymon Wąsowicz

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### On error bounds for Gauss–Legendre and Lobatto quadrature rules

For a function  $f: [-1, 1] \rightarrow \mathbb{R}$  let

$$\begin{aligned}\mathcal{G}_2(f) &:= \frac{1}{2} \left( f \left( -\frac{\sqrt{3}}{3} \right) + f \left( \frac{\sqrt{3}}{3} \right) \right), \\ \mathcal{G}_3(f) &:= \frac{5}{18} f \left( -\frac{\sqrt{15}}{5} \right) + \frac{4}{9} f(0) + \frac{5}{18} f \left( \frac{\sqrt{15}}{5} \right), \\ \mathcal{L}(f) &:= \frac{1}{12} f(-1) + \frac{5}{12} f \left( -\frac{\sqrt{5}}{5} \right) + \frac{5}{12} f \left( \frac{\sqrt{5}}{5} \right) + \frac{1}{12} f(1), \\ \mathcal{S}(f) &:= \frac{1}{6} (f(-1) + 4f(0) + f(1)).\end{aligned}$$

The operators  $\mathcal{G}_2$  and  $\mathcal{G}_3$  are connected with Gauss–Legendre quadrature rules. The operators  $\mathcal{L}$  and  $\mathcal{S}$  are connected with Lobatto and Simpson’s quadrature rules, respectively. We establish the following inequalities:

#### PROPOSITION 1

If  $f: [-1, 1] \rightarrow \mathbb{R}$  is 3-convex, then  $\mathcal{G}_2(f) \leq \mathcal{G}_3(f) \leq \mathcal{S}(f)$  and  $\mathcal{L}(f) \leq \mathcal{S}(f)$ .

We apply this result to give the error bounds for quadrature rules  $\mathcal{G}_3$  and  $\mathcal{L}$  for four times differentiable functions (instead of six times differentiable functions as in the classical results known from numerical analysis).

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# Marek Żołądak

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## Asymptotic stability of isometries in $p$ -homogeneous F-spaces

Joint work with Józef Tabor.

The equation of isometry in Banach spaces is stable in Hyers-Ulam sense. It happens that in complete Frechet spaces this equation is not stable. We prove that if  $p \in (0, 1]$ ,  $r > 0$ ,  $\varepsilon > 0$ ;  $X, Y$  are complete  $p$ -homogeneous spaces,  $f: X \rightarrow Y$  is a surjective mapping such that  $f(0) = 0$  and

$$| \|f(x) - f(y)\|^r - \|x - y\|^r | \leq \varepsilon \quad \text{for } x, y \in X,$$

then there exist a linear surjective isometry  $U$  and a constant  $L > 0$  such that

$$\|f(x) - U(x)\| \leq L(\varepsilon^p \|x\|^{1-pr} + \varepsilon^{\frac{1}{r}}) \quad \text{for } x \in X,$$

when  $pr < 1$ , and

$$\|f(x) - U(x)\| \leq L\varepsilon^{\frac{1}{r}} \quad \text{for } x \in X,$$

when  $pr > 1$ .