ON SOME PROBLEMS CONCERNING A SUM TYPE OPERATOR

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The sum type operator F, given by

$$F[\varphi](x) := \sum_{k=0}^{\infty} \frac{1}{2^k} \varphi(2^k x)$$

has been thoroughly discussed in the last years, see [1] - [8]. Nevertheless there remained some open problems. We state here a few of them which are connected to five aspects of the operator F.

1. Images and pre-images of F

To keep things simple, let us consider F just on the domain

$\mathcal{C} := \{\varphi : \mathbb{R} \to \mathbb{R}\ ;\ \varphi$ continuous and 1-periodic$\}$.

Then F is a Banach space automorphism. The functions w and $t,$ given by $w(x) = cos 2\pi x,\ t(x) = dist(x, \mathbb{Z})$, belong to the set $\mathcal{C}$ and generate the prominent Weierstrass resp. Takagi functions $F[w]$ and $F[t]$, both continuous and nowhere differentiable (cnd). Let

$\mathcal{N} := \{f : \mathbb{R} \to \mathbb{R}\ ;\ f$ cnd and 1-periodic$\}$.

P1. *Characterize* $\mathcal{M} := F^{-1}[\mathcal{N}] = \{\varphi \in \mathcal{C}\ ;\ F[\varphi] \in \mathcal{N}\}$.

There are some subsets of $\mathcal{M}$ which are interesting by themselves.

P2. *Characterize* $\mathcal{M}_1 := F^{-1}[\mathcal{N}] \cap \mathcal{N} = \{\varphi \in \mathcal{M}\ ;\ \varphi\ cnd\}$.

P3. *Characterize* $\mathcal{M}_2 := \{\varphi \in \mathcal{M}\ ;\ \varphi\ polygonal\}$.

P4. *Characterize* $\mathcal{M}_3 := \{\varphi \in \mathcal{M}\ ;\ \varphi\ convex\ on\ [0,1]\}$.

The above mentioned t belongs to $\mathcal{M}_2$ and to $\mathcal{M}_3$. Some partial answers to **P1. - P4.** are contained in [2] and [3].

In [3] we discussed $\varphi_v \in \mathcal{C}$ which are generated by a real sequence $v = (v_n)$:

$$\varphi_v(x) := \sum_{n=0}^{\infty} \frac{v_n}{2^n}\ cos(2\pi 2^n x)$$

and found sufficient conditions for v to guarantee that $\varphi_v \in \mathcal{M}_1$:

[3], **Theorem 2.** *Let* (v_n) *be a bounded sequence of positive real numbers which does not converge to zero. Then* $\varphi_v \in \mathcal{N}$ *and* $F[\varphi_v] \in \mathcal{N}$. –

P5. *Characterize* $\mathcal{V} := \{(v_n)\ ;\ \varphi_v \in \mathcal{M}_1\}$.

[3] contains also some partial information regarding

P6. *Characterize* $\mathcal{M}_4 := \{\varphi \in \mathcal{C}\ ;\ F^m[\varphi] \in \mathcal{N}\ for\ every\ m \in \mathbb{N}\}$.

2. Spectral properties of F

Spectral properties of F have been investigated in [4], [5], [6], [7] and [8].
In [7] and [8], the operator F has been considered on sixteen different domains, all of them vector spaces. In [7], joint work with K. Baron, all the according eigenspaces have been described. One arithmetical problem remained open.
Let $K_\alpha := \alpha \cdot 2^{\mathbb{Z}} + \mathbb{D}$ ($\mathbb{Z}$ integers, $\mathbb{D}$ dyadic rationals).

P7. *Characterize* $S := \{\alpha \in \mathbb{R}\ ;\ K_\alpha = K_{-\alpha}\}$.

This problem was already stated in Studia Math. **5** (2006), p. 118. More information is provided in [7], p. 258.
In [8], all the according continuous and residual spectra have been calculated except in two cases. Let $\mathcal{C}$ as in **1.** and $\mathcal{C}' := \{\varphi \in \mathcal{C}\ ;\ \varphi \text{ even}\}$.

P8. *Compute the continuous and the residual spectra of the Banach space automorphisms* $F : \mathcal{C} \to \mathcal{C}$ and $F : \mathcal{C}' \to \mathcal{C}'$.

3. The maximal domain of F

$$\mathcal{D} := \{\varphi : \mathbb{R}\ ;\ \sum_{k=0}^{\infty} \frac{1}{2^k}\ \varphi(2^k x) \text{ converges for every } x \in \mathbb{R}\ \}$$

is the maximal possible domain of our operator with respect to arguments $\varphi : \mathbb{R} \to \mathbb{R}$. $\mathcal{D}$ is a real vector space containing the following subsets $\big(J := (-2,-1] \cup [1,2)\big)$:

$$\mathcal{D}_A := \{\varphi : \mathbb{R} \to \mathbb{R}\ ;\ \exists\ a : \mathbb{R} \to \mathbb{R}, a \text{ additive} : \varphi(x) = a\ (\log|x|)\}$$

$$\mathcal{D}_B := \{\varphi : \mathbb{R} \to \mathbb{R}\ ; \exists\ b : J \to \mathbb{R}\ :\ \varphi = b \text{ on } J, \varphi(x) = \varphi(2x) \text{ otherwise}\}$$

Both subsets are far from exhausting the full set $\mathcal{D}$. This can be seen from [6], joint work with P. Volkmann, where the set $\mathcal{D}$ is characterized:

[6], **Satz 2.** *Equivalent are*

(I) $\varphi \in \mathcal{D}$,

(II) $\forall t \in J \ \exists (a^t_\mu)_{\mu\in\mathbb{Z}} \ : \ \sum_{k=0}^{\infty} a^t_k \ \in \mathbb{R}$ *and*

$\forall x \in \mathbb{R} \setminus \{0\} \ ; \ \varphi(x) = \varphi(2^{m(x)}\tau(x)) = 2^{m(x)} a^{\tau(x)}_{m(x)}$. –

(II) can be replaced by the equivalent statement

(III) $\exists a_\mu : J \to \mathbb{R} \ (\mu \in \mathbb{Z}) \ \forall \xi \in J : \sum_{k=0}^{\infty} a_k(\xi) \in \mathbb{R}$ *and*

$\forall \xi \in J \ , \ m \in \mathbb{Z} \ : \ \varphi(2^m \xi) = 2^m a_m(\xi)$,

which allows a straightforward comparision of $\mathcal{D}$ and $\mathcal{D}_B$. In fact, $\mathcal{D}_B$ is the eigenspace of $F : \mathcal{D} \to F[\mathcal{D}]$ with respect to the eigenvalue 2 and can be rewritten as

$\mathcal{D}_{\mathcal{B}} = \{\varphi : \mathbb{R} \to \mathbb{R} \ ; \ \forall x \in \mathbb{R} \ : \ \varphi(x) = \varphi(2x)\}$.

This last representation suggests the following

P9. *Characterize $\mathcal{D}$ as solution set of a system of (iterative) functional equations.*

Connected with the above Satz 2 is

P10. *Which functions $a_\mu : J \to \mathbb{R}$ $(\mu \in \mathbb{Z})$ generate $\varphi \in \mathcal{A}$?*

$\mathcal{A} \subset \mathcal{D}$ can be any relevant function set, but we have especially in mind sets like $\mathcal{A}_1 = \mathcal{D}_{\kappa,\varrho} \cap \mathcal{N}^*$ or $\mathcal{A}_2 = F^{-1}[\mathcal{N}^*]$.
Here we used the notation from [8] for the sixteen subspaces $\mathcal{D}_{\kappa,\varrho} (1 \le \kappa, \varrho \le 4)$ of $\mathcal{D}$ already mentioned in **2**. $\mathcal{N}^*$ may be any suitable set of cnd functions, whose elements must not necessarily be 1-periodic or bounded (as the elements of $\mathcal{N}$ are). **P10.** with $\mathcal{A} = \mathcal{A}_2$ is closely related to the problems from Section **1**.

4. Characterizations of $F[\varphi]$

Assume that $\varphi \in \mathcal{C}'$, i.e. , φ is bounded, 1-periodic and even. Then $F[\varphi]$ satisfies the following system of functional equations

(1) $f(x) - 2f(\frac{x}{2}) = -2\varphi(\frac{x}{2})$,

(2) $f(x) - 2f(\frac{x+1}{2}) = -2\varphi(\frac{x+1}{2})$,

(3) $f(\frac{x}{2}) - f(\frac{x+1}{2}) = \varphi(\frac{x}{2}) - \varphi(\frac{x+1}{2})$,

(4) $f(x) - f(\frac{x}{2}) - f(\frac{x+1}{2}) = -\varphi(\frac{x}{2}) - \varphi(\frac{x+1}{2})$,

(5) $f(x+1) - f(x) = 0$,

(6) $f(-x) - f(x) = 0$,

(7) $f(1-x) - f(x) = 0$.

The paper [1] contains a thorough investigation of this and the associated homogeneous system. Moreover all possible characterizations of $F[\varphi]$ by subsystems of (1) - (7) in the frame of boundedness and of continuity are given:

[1], **Theorem 2.** *Assume that $f : \mathbb{R} \to \mathbb{R}$ is bounded and $\varphi \in \mathcal{C}'$. Then $f = F[\varphi]$ iff either (1) or (2) or (3, 4) holds. Any "superset" of either (1) or (2) or (3,4) is characteristic as well. No other "subset" of (1) - (7) is characteristic.* -

[1], **Theorem 3.** *Assume that $f : \mathbb{R} \to \mathbb{R}$ is continuous and $\varphi \in \mathcal{C}'$. Then $f = F[\varphi]$ iff either (1,2) or (1,3) or (1,4) or (1,5) or (1,7) or (2,3) or (2,4) or (2,5) or (2,6) or (2,7) or (3,4) holds. Any "superset" of either (1,2) or ... or (3,4) is characteristic as well. No other "subset" of (1) - (7) is characteristic.* -

In the language of [8] we have $\varphi \in \mathcal{D}_{3,4}$ in the above Theorems. But there are in fact sixteen possible cases $\varphi \in \mathcal{D}_{\kappa,\varrho}$ - in some of them only a "subset" of (5), (6), (7) is available - and many more possible regularity assumptions on f. This leads to

P11. *Give a list of all possible characterizations of $F[\varphi], \varphi \in \mathcal{D}_{\kappa,\varrho}$, under suitable regularity assumptions on f.*

To find out those properties of φ resp. f, which give interesting modifications (extensions) of Theorems 2 and 3 above, is left to the reader. Some partial results are already given in [1].

5. A two parameter extension of F

Let $\alpha \in (0,1), \beta \in (0,\infty)$ and $\mathcal{B} := \{\varphi : \mathbb{R} \to \mathbb{R} \ ; \ \varphi \text{ bounded}\}$.
Then

$$F_{\alpha,\beta} : \mathcal{B} \to \mathcal{B}, \quad F_{\alpha,\beta}[\varphi](x) := \sum_{k=0}^{\infty} \alpha^k \varphi(\beta^k x)$$

provides a two parameter extension of F : $F = F_{\frac{1}{2},2}$.

In the paper [4], $F_{\alpha,\beta}$ is defined on the Banach space $\mathcal{B}$ and on five closed subspaces $\mathcal{B}_1$ to $\mathcal{B}_5$. It is proved that $F_{\alpha,\beta} : \mathcal{B} \to \mathcal{B}$ is a Banach space automorphism; the operator norms of $F_{\alpha,\beta}$ and $F_{\alpha,\beta}^{-1}$ are computed and the point spectra and eigenspaces of $F_{\alpha,\beta}$ and $F_{\alpha,\beta}^{-1}$ are given. Similar results for $F_{\alpha,\beta} : \mathcal{B}_n \to \mathcal{B}_n (1 \leq n \leq 5)$ are obtained, but only for $\beta \in \mathbb{N}\backslash\{1\}$. Moreover, there is no complete description of the eigenspaces in the spirit of [7] and no information on the continuous and residual spectra. The paper [4] closes with stating several problems concerning images and pre-images of $F_{\alpha,\beta}$, as we did in Section **1**. for F. Instead of formulating $\alpha, \beta-$ modifications of our Problems **1 - 11**, we rather present a final project

P12. *Discuss in connection with* $F_{\alpha,\beta}$

- *Images and pre-images,*
- *Eigenspaces, continuous and residual spectra,*
- *Fourier series,*
- *Maximal domains,*
- *Characterizations,*
- *Identifications.*

Fourier series properties could extend results of [2] and [3]. The last point "Identifications" takes care of the fact that particular properties of $F_{\alpha,\beta}[\varphi]$ may essentially depend on α and β.
Example: Identify all α, β such that $F_{\alpha,\beta}[w]$ is cnd.
Recall that $F_{\alpha,\beta}[w](x) = \sum_{k=0}^{\infty} \alpha^k \; cos\ (2\pi\beta^k x)$.
A famous result of G.H. Hardy gives a concise solution:

$F_{\alpha,\beta}[w]$ is cnd if and only if $\alpha\beta \geq 1$.

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