POLAND, BĘDLEWO 2023
20th ICFEI

List of Talks

1. Dennis Epebinu Abayomi, *Inequalities for fractional integrals with use of stochastic ordering*
2. Ulrich Abel, *Characterizing convexity of functions by inequalities of Bernstein polynomials*
3. Ana-Maria Acu, *Inequalities for Aldaz-Kounchev-Render operators*
4. Laddawan Aiemsomboon, *Fixed Point Approach to Stability of an Additive-Quadratic Functional Equation on Orthogonality in the sense of Rätz*
5. Roman Badora, *On a separation of two functionals by an additive map*
6. Karol Baron, *Around Hölder continuous solutions of an iterative equation*
7. Pál Burai, *Limit theorems for Bajraktarević and Cauchy quotient means of independent identically distributed random variables*
8. Jacek Chmieliński, *On continuity of norms*
9. Jacek Chudziak, **On representations of the certainty equivalents for binary lotteries**
10. Farzad Dadipour, *From the triangle inequality to characterization of p-Banach spaces*
11. Beata Deręgowska, *On maximal projection constants*
12. Włodzimierz Fechner, *Some remarks on the Casino Inequality of Dubins and Savage*
13. Gian Luigi Forti, *Some classes of C^1 solutions of an alternative Cauchy functional equation in three unknown functions.*
14. Souvik Ghosh, *On some geometric constants in a Banach space*
15. Dorota Głazowska, *Subcommutativity of integrals similar to integral quasi-arithmetic means*
16. Chaitanya Gopalakrishna, *On the nonexistence of iterative roots of functions*
17. Szymon Ignaciuk, *Relationship between some angular inequalities and Kaplan classes for complex polynomials*
18. Justyna Jarczyk, *On a linear iterative equation of finite order and its solutions with a prescribed asymptotics*
19. Witold Jarczyk, *Iterative roots of piecewise monotonic functions extended from the characteristic interval*
20. Divya Khurana, *Birkhoff-James orthogonality and its pointwise symmetry in some Banach spaces*

21. Tibor Kiss, *Some solutions of the Járai–Maksa–Páles functional equation*
22. Milica Klaričić Bakula, *Jensen-Steffensen Inequality: Accentuate the Negative*
23. Paweł A. Kluza, *Inequalities for Jensen–Sharma–Mittal and Jeffreys–Sharma–Mittal type f–divergences*
24. Tomasz Kochanek, **Approximate multiplicativity and zero-product preserving in Banach algebras**
25. Dawid Komorek, *Continuous dependence of the weak limit of iterates of some random-valued vector functions*
26. Zbigniew Leśniak, *On foliations consisting of invariant lines of a Brouwer homeomorphism*
27. Joanna Markowicz, *García-Falset coefficient and Opial property in direct sums of Banach spaces*
28. Kazuki Okamura, *Properties of quasi-arithmetic means of random variables*
29. Andrzej Olbryś, *On approximate convexity*
30. Adam J. Ostaszewski, *Homomorphisms from Functional Equations: The Goldie Equation Revisited*
31. Zdzisław Otachel, *Jensen-Jessen inequality for sublinear functionals*
32. Paweł Pasteczka, *Invariance property for extended means*
33. Kallol Paul, **Geometry of the space of linear operators through the lens of Birkhoff-James orthogonality**
34. Teresa Rajba, *On the Raşa, Hermite-Hadamard and Jensen inequalities for Popoviciu's box-(m,n)-convex functions*
35. Ioan Raşa, *Kantorovich type operators and convexity*
36. Jens Schwaiger, **The Sincov equation, history, applications, and new developments**
37. Wutiphol Sintunavarat, **The modification of fuzzy numbers with its impact on the stability of fuzzy number-valued functional equations**
38. Aleksandra Świątczak, *Functional equations characterizing differential operators*
39. László Székelyhidi, *New Results on Spectral Synthesis*
40. Patricia Szokol, *G-majorization and Schur-type convexity*
41. Bettina Wilkens, *An alternative additive-quadratic equation*
42. Sebastian Wójcik, *Zero utility principle under uncertainty*
43. Marek Cezary Zdun, *On Koopmans recursion*

Dennis Epebinu Abayomi
University of Silesia, Katowice, Poland

Inequalities for fractional integrals with use of stochastic ordering

In this talk, we show the connections of inequalities involving fractional integrals with Ohlin lemma and Levin-Stechkin theorem. First, we give a new proof of the fractional version of the inequality of Hermite Hadamard type and then we extend it in two directions. Thus we compare the fractional expression occurring in this inequality with the usual integral and we obtain stronger inequalities (one of them is related to Bullen inequality).

Ulrich Abel

Technische Hochschule Mittelhessen, Friedberg, Germany

Characterizing convexity of functions by inequalities of Bernstein polynomials

(joint work with Dany Leviatan and Ioan Raşa)

About 30 years ago Ioan Raşa stated the following inequality involving Bernstein basis polynomials $p_{n,\nu}(x) = \binom{n}{\nu} x^{\nu} (1-x)^{n-\nu}$ and convex functions as an open problem:

Problem. *Prove or disprove:*
Let $n \in \mathbb{N}$. *If* $f \in C[0,1]$ *is a convex function, then, for all* $x, y \in [0,1]$,

$$\sum_{i=0}^{n} \sum_{j=0}^{n} \left[p_{n,i}(x) p_{n,j}(x) + p_{n,i}(y) p_{n,j}(y) - 2 p_{n,i}(x) p_{n,j}(y) \right] f\left(\frac{i+j}{2n}\right) \geq 0.$$

In 2017, Mrowiec, Rajba and Wąsowicz [5] affirmed the conjecture in positive. Their proof makes heavy use of probability theory. A short elementary proof of the above inequality and the corresponding results for Mirakyan–Favard–Szász operators and Baskakov operators can be found in [1].
In 2020, Abel and Leviatan [2] found an inequality generalizing the above one to q-monotone functions on $[0,1]$, for any positive integer q. In 2022, they and Raşa [3] proved a converse for functions in $C[0,1]$. The combination of both results provides a characterization of q-monotone functions in $C[0,1]$.

Finally, we study the preservation of q-monotonicity of various Durrmeyer-type operators in $[0,1]$ or $[0,\infty)$ as the case may be. These results yield further characterizations of continuous q-monotone functions.

References

[1] Abel, Ulrich. "An inequality involving Bernstein polynomials and convex functions." *J. Approx. Theory* 222 (2017): 1-7.

[2] Abel, Ulrich, and Dany Leviatan. "An extension of Raşa's conjecture to q-monotone functions." *Results Math.* 75, no. 4 (2020): Paper No. 180.

[3] Abel, Ulrich, Dany Leviatan, and Ioan Raşa. "Relations between the Bernstein polynomials and q-monotone functions." *Results Math.* 77, no. 6 (2022): Paper No. 239.

[4] Abel, Ulrich, and Ioan Raşa. "A sharpening of a problem on Bernstein polynomials and convex functions." *Math. Inequal. Appl.* 21, no. 3 (2018): 773-777.

[5] Mrowiec, Jacek, Teresa Rajba, and Szymon Wąsowicz. "A solution to the problem of Raşa connected with Bernstein polynomials." *J. Math. Anal. Appl.* 446, no. 1 (2017): 864-878.

Ana-Maria Acu
Lucian Blaga University of Sibiu, Romania

Inequalities for Aldaz-Kounchev-Render operators

The Aldaz-Kounchev-Render operators are positive linear operators on $C[0,1]$ which preserve simultaneously the functions $\mathbf{1}$ and x^j for a given integer $j \geq 2$. Recently, this type of operators have been considered on finite-dimensional hypercubes and simplices; see [1], [2], [3], [4]. Besides the approximation properties we investigate their behavior with respect to the convex functions.

References

[1] Abel, Ulrich, Ana Maria Acu, Margareta Heilmann, and Ioan Raşa. "Genuine Bernstein-Durrmeyer type operators preserving 1 and x^j", submitted.

[2] Acu, Ana-Maria, Stefano De Marchi, and Ioan Raşa. "Aldaz-Kounchev-Render operators and their approximation properties." *Results Math.* 78, no. 1 (2023): Paper No. 21.

[3] Acu, Ana-Maria, and Ioan Raşa. "An asymptotic formula for Aldaz-Kounchev-Render operators on the hypercube." *Arxiv* (2023): arxiv.org/pdf/2301.01280.pdf

[4] Acu, Ana-Maria, Stefano De Marchi, and Ioan Raşa. "Aldaz-Kounchev-Render operators on simplices." submitted.

Laddawan Aiemsomboon

Thammasat University, Phatumthani, Thailand

Fixed Point Approach to Stability of an Additive-Quadratic Functional Equation on Orthogonality in the sense of Rätz

(joint work with Wutiphol Sintunavarat and Anurak Thanyacharoen)

Let $(X, \perp)$ be an orthogonality Banach space in the sense of Rätz and Y be a Banach space. In this talk, we show applictions of the alternative fixed point theorem for proving the Hyers-Ulam stability of the generalized orthogonally additive-quadratic functional equation of the form

$$f(ax+by)+f(ax-by)+2b^2f(y)=(a^2+a)f(x)+(a^2-a)f(-x)+b^2f(2y)$$

for all $x, y \in X$ with $x \perp y$, where a, b are non-zero rational numbers and $f\colon X \to Y$ is an unknown function.

Roman Badora

Uniwersytet Śląski, Katowice, Poland

On a separation of two functionals by an additive map

In the talk we will deal with the problem of separating two functionals defined on a semigroup by an additive map. We examine whether the fulfillment of a given system of inequalities by two functionals guarantees the existence of an additive map separating these functionals. The classical separation theorem for subadditive and superadditive functionals provides an example of a system of inequalities guaranteeing separation by an additive map. In the talk we answer the question of how it is for other systems of inequalities.

Karol Baron
University of Silesia, Katowice, Poland

Around Hölder continuous solutions of an iterative equation

Assume that (X,ρ) is a complete and separable metric space, $(\Omega,\mathcal{A},P)$ is a probability space, $f\colon X\times\Omega\to X$ is measurable for $\mathcal{B}\otimes\mathcal{A}$, where $\mathcal{B}$ denotes the σ–algebra of all Borel subsets of X,

$$\int_\Omega \rho\big(f(x,\omega),f(z,\omega)\big)P(d\omega)\le\lambda\rho(x,z)\quad\text{for } x,z\in X$$

with a $\lambda\in(0,1)$, and

$$\int_\Omega \rho\big(f(x,\omega),x\big)P(d\omega)<\infty\qquad\text{for } x\in X.$$

Given $F\colon X\to\mathbb{R}$ we consider the problem of the existence and uniqueness of solutions $\varphi\colon X\to\mathbb{R}$ of the equation

$$\varphi(x)=\int_\Omega\varphi\big(f(x,\omega)\big)P(d\omega)+F(x)$$

such that $\varphi-F$ is Hölder continuous.

Pál Burai

Budapest University of Technology and Economics, Hungary

Limit theorems for Bajraktarević and Cauchy quotient means of independent identically distributed random variables

(joint work with Mátyás Barczy)

We derive strong law of large numbers and central limit theorems for Bajraktarević, Gini and exponential- (also called Beta-type) and logarithmic Cauchy quotient means of independent identically distributed (i.i.d.) random variables. The exponential- and logarithmic Cauchy quotient means of a sequence of i.i.d. random variables behave asymptotically normal with the usual square root scaling just like the geometric means of the given random variables. Somewhat surprisingly, the multiplicative Cauchy quotient means of i.i.d. random variables behave asymptotically in a rather different way: in order to get a non-trivial normal limit distribution a time dependent centering is needed.

Jacek Chmieliński

Pedagogical University of Krakow, Poland

On continuity of norms

In my talk given at the 19th ICFEI the results published in [1] were presented. Continuing this topic, we will present some new relationships between two norms on an infinite-dimensional vector space X. We consider, in particular, the unilateral continuity of one norm with respect to the other, answering the two questions:

1. Does any norm N in X admit another norm N' in X such that N is continuous with respect to N' but N' is discontinuous with respect to N?
2. Does any norm N in X admit a norm N' in X such that N' is continuous with respect to N but N is discontinuous with respect to N'?

References

[1] Chmieliński, Jacek, and Moshe Goldberg. "Continuity and discontinuity of seminorms on infinite-dimensional vector spaces (I and II)." *Linear Algebra Appl.* 578 (2019): 153-158; 594 (2020): 249-261.

Jacek Chudziak
University of Rzeszów, Poland

On representations of the certainty equivalents for binary lotteries

(joint work with Michał Lewandowski)

Let $(x, y; p)$ denote a binary monetary lottery that pays x with probability p, and y with probability $1-p$. Assume that $F(x, y; p)$ is the certainty equivalent of $(x, y; p)$, i.e. a certain amount, the receipt of which is as good for the decision maker as playing the lottery. We are interested in individual preferences that lead to the following representation of the certainty equivalent

$$F(x, y; p) = u^{-1}\Big(w(p)u(x) + \big(1 - w(p)\big)u(y)\Big), \tag{1}$$

where $u\colon \mathbb{R} \to \mathbb{R}$ is a continuous strictly increasing utility function and $w\colon [0,1] \to [0,1]$ is a probability weighting function. We consider several axioms leading to (1). In particular, we characterize (1) in the domain of all binary lotteries (arbitrary payouts and probabilities) as well as in the subset of simple lotteries, where one of the two payouts of a lottery is fixed. For both domains we deal with the case of the general w and the case where w is the identity function. The latter case corresponds to the certainty equivalent of a binary lottery under the expected utility theory. In addition, for the general w, we separately characterize the case where $F(x, y; p)$ depends on the rank of the payouts (rank dependency) and the case where it does not.

Farzad Dadipour
Graduate University of Advanced Technology, Kerman, Iran

From the triangle inequality to characterization of p-Banach spaces

(joint work with Farzane Sadeghi Boome)

In this talk, we deal with the generalized triangle inequality of the second type and its reverse in the framework of quasi Banach spaces. More precisely, by using the well-known Aoki-Rolewicz theorem and the concept of equivalent p-norms, we provide some necessary and sufficient conditions for n-tuples to satisfy the mentioned inequalities.

As applications, we improve some already known results and present some characterizations of p-Banach spaces among quasi Banach spaces. In particular, we show that a quasi Banach space such as X is a p-Banach space if and only if for all $(\mu_1, ..., \mu_n) \in \mathbb{R}^n$ satisfying $\mu_j > 0$ for some j and $\mu_i < 0$ for all $i \neq j$, the reverse of the generalized triangle inequality of the second type

$$\sum_{i=1}^{n} \frac{\|x_i\|^p}{\mu_i} \leq \left\| \sum_{i=1}^{n} x_i \right\|^p \qquad \text{(where } x_i \in X\text{)}$$

holds only with the assumption $\mu_j \geq \max_{i\in\{1,...,n\}-\{j\}}\{1, |\mu_i|\}$.

References

[1] Dadipour, Farzad, and Asiyeh Rezaei. "Characterization of p-Banach Spaces Based on a Reverse Triangle Inequality." *Bull. Malays. Math. Sci. Soc.* 46, no. 5 (2023): Art. No. 173.
[2] Dadipour, Farzad, Mohammad Sal Moslehian, John Michael Rassias, and Sin-Ei Takahasi. "Characterization of a generalized triangle inequality in normed spaces." *Nonlinear Anal.* 75, no. 2 (2012): 735-741.
[3] Izumida, Tamotsu, Ken-Ichi Mitani, and Kichi-Suke Saito. "Another approch to characterizations of generalized triangle inequalities in normed spaces." *Open Math.* 12, no. 11 (2014): 1615-1623.
[4] Rezaei, Asiyeh, and Farzad Dadipour. "Generalized triangle inequality of the second type in quasi normed spaces." *Math. Inequal. Appl.* 23, no. 4 (2020): 1155-1163.
[5] Saitoh, Saburou. "Generalizations of the triangle inequality." *JIPAM. J. Inequal. Pure Appl. Math.* 4, no. 3 (2003): Article 62.

Beata Deręgowska
Pedagogical University of Krakow, Poland

On maximal projection constants

Let X be a Banach space over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Let $Y \subseteq X$ be a subspace. By $\mathcal{P}(X,Y)$ we denote the set of all linear and continuous projections from X onto Y, recalling that an operator $P\colon X \to Y$ is called a *projection* onto Y if $P|_Y = \mathrm{Id}_Y$. We define the *relative projection constant of subspace* Y of space X by

$$\lambda(Y,X) := \inf\big\{\|P\| : \ P \in \mathcal{P}(X,Y)\big\}.$$

Now we can define the *absolute projection constant of* Y by

$$\lambda(Y) := \sup\big\{\lambda(Y,X) : Y \subseteq X\big\}.$$

The ultimate goal of researchers in this area is to determine the exact value of *maximal absolute projection constant*, which is defined by

$$\lambda_{\mathbb{K}}(m) := \sup\big\{\lambda(Y) : \ \dim(Y) = m\big\}.$$

In 1994, H. König and N. Tomaczak-Jaegermann stated the following estimation.

Theorem 1 (stated in [3]; proved in [1]). *Let $m > 1$ then*

1. $\lambda_{\mathbb{R}}(m) \leq \frac{2}{m+1}\big(1 + \frac{m-1}{2}\sqrt{m+2}\big)$,
2. $\lambda_{\mathbb{C}}(m) \leq \frac{1}{m}\big(1 + (m-1)\sqrt{m+1}\big)$.

Unfortunately, their proof is based on an erroneous lemma, as was pointed out in [2]. In this talk, we will present the correct proof of the latter. Moreover relying on this result we provide the exact values of $\lambda_{\mathbb{K}}(m)$ in the cases where the maximal equiangular tight frame exists in $\mathbb{K}^m$.

References

[1] Deręgowska, Beata, and Barbara Lewandowska. "A simple proof of the Grünbaum conjecture." *J. Funct. Anal.* 285, no. 2 (2023): Paper No. 109950.

[2] Chalmers, Bruce Langworthy, and Grzegorz Lewicki. "Three-dimensional subspace of $l_\infty^{(5)}$ with maximal projection constant." *J. Funct. Anal.* 257, no. 2 (2009): 553-592.

[3] König, Hermann, and Nicole Tomczak-Jaegermann. "Norms of minimal projections." *J. Funct. Anal.* 119, no. 2 (1994): 253-280.

Włodzimierz Fechner
Lodz University of Technology, Poland

Some remarks on the Casino Inequality of Dubins and Savage

L.E. Dubins and L.J. Savage in their seminal book *How to gamble if you must. Inequalities for stochastic processes* [1] provided formal definitions of many fundamental notions of gambling, like gambles and lotteries, gambling house (as a family of gambles available for the gambler at a given moment) and a casino (a special type of a gambling house). They introduced the *casino inequality*:

$$U\big(xy + (1-x)z\big) \geq U(x)U(y) + \big(1 - U(x)\big)U(z), \tag{1}$$

with $x \in [0,1]$, $0 \leq z \leq y \leq 1$, where $U\colon [0,\infty) \to [0,1]$ satisfies $U(0) = 0$ and $U(x) = 1$ for $x \geq 1$. The function U represents the optimal probability of victory in a casino of a fixed goal of 1 (i.e. of reaching 1 starting with a given fortune) and x, y, z are possible fortunes. The fundamental theorem [1, Theorem 4.2.1, p. 64] establishes an equivalence between solutions of (1) and optimal probabilities of attaining a fixed goal in a casino.

During the talk we collect fundamental properties of the inequality (1). Further, we exploit its connections with other functional inequalities, in particular with the Hosszú inequality.

References

[1] Dubins, Lester Eli, and Leonard Jimmie Savage. *How to gamble if you must. Inequalities for stochastic processes.* New York-Toronto-London-Sydney: McGraw-Hill Book Co., 1965.

Gian Luigi Forti
Università degli Studi di Milano, Italy

Some classes of C^1 solutions of an alternative Cauchy functional equation in three unknown functions.

During ICFEI 19 it has been proved that the if the functions f, g, h are real analytic solutions of the alternative Cauchy equation

$$\big[f(x+y)-f(x)-f(y)\big]\big[g(x+y)-g(x)-g(y)\big]\big[h(x+y)-h(x)-h(y)\big]=0,$$

then at least one of them must be linear (see [1]). The natural open question is to find a way to construct less regular solutions of the previous equation. By using the integral representation of the Cauchy difference, some classes of C^1 solutions of the alternative Cauchy functional equation are presented. Then, some open problems are presented.

References

[1] Forti, Gian Luigi. "Alternative Cauchy equation in three unknown functions." *Aequationes Math.* 95, no. 6 (2021): 1233-1242.

Souvik Ghosh
Jadavpur University, Kolkata, India

On some geometric constants in a Banach space

In this talk we describe how the notion of isosceles orthogonality plays a crucial role regarding the attainment of the James constant and the Schäffer constant. We further study the attainment of the James constant in a two-dimensional polyhedral Banach space $\mathbb{X}$. We show that there exists an extreme point x on the unit ball of $\mathbb{X}$ such that the James constant attains its value at x.

We also study the approximate isosceles orthogonality and find some geometrical structures associated with it. Moreover, we establish the relation between the approximate isosceles orthogonality and the notion of modulus of convexity in a Banach space.

This talk is based on the article [1].

References

[1] Sain, Debmalya, Souvik Ghosh, and Kallol Paul. "On isosceles orthogonality and some geometric constants in a normed space." *Aequationes Math.* 97, no. 1 (2023): 147-160.

Dorota Głazowska
University of Zielona Góra, Poland

Subcommutativity of integrals similar to integral quasi-arithmetic means

(joint work with Paolo Leonetti, Janusz Matkowski and Salvatore Tringali)

Let $(X, \mathcal{L}, \lambda)$ and $(Y, \mathcal{M}, \mu)$ be finite non-degenerate measure spaces, $I \subseteq \mathbb{R}$ be a non-empty interval, and $f, g\colon I \to \mathbb{R}$ be continuous injections. We look for conditions on f and g under which the inequality

$$f^{-1}\left(\int_X f\left(g^{-1}\left(\int_Y g \circ h \, d\mu\right)\right) d\lambda\right) \leq g^{-1}\left(\int_Y g\left(f^{-1}\left(\int_X f \circ h \, d\lambda\right)\right) d\mu\right) \quad (1)$$

is satisfied for every $h\colon X \times Y \to I$ in a suitable class of $\mathcal{L} \otimes \mathcal{M}$-measurable simple functions, taking for granted that each side of the above inequality is well defined. Notice that setting

$$\mathcal{M}_f(h) = f^{-1}\left(\int_X f \circ h \, d\lambda\right), \qquad \mathcal{M}_g(h) = g^{-1}\left(\int_Y g \circ h \, d\mu\right)$$

for the $\mathcal{L} \otimes \mathcal{M}$-measurable simple functions $h\colon X \times Y \to I$, the inequality (1) holds if

$$\mathcal{M}_f \circ \mathcal{M}_g \leq \mathcal{M}_g \circ \mathcal{M}_f,$$

and it can be interpreted as the "subcommutativity of the pair $(\mathcal{M}_f, \mathcal{M}_g)$" or "supercommutativity of the pair $(\mathcal{M}_g, \mathcal{M}_f)$".
Moreover, if $(X, \mathcal{L}, \lambda)$ and $(Y, \mathcal{M}, \mu)$ are probability spaces and the image $h(X \times Y)$ is contained in a compact subset of I for every test function h, which is especially the case when $h\colon X \times Y \to I$ is an $\mathcal{L} \otimes \mathcal{M}$-measurable simple function, then each side of the inequality (1) is well posed and can be interpreted as "partially mixed integral quasi-arithmetic means".

Chaitanya Gopalakrishna
Indian Statistical Institute, Bangalore, India

On the nonexistence of iterative roots of functions
(joint work with B. V. Rajarama Bhat)

An *iterative root of order* $n \geq 2$ of a self-map f on a nonempty set X is a self-map g on X such that $g^n = f$. We discuss a new result on the nonexistence of iterative roots of self-maps on arbitrary sets and use it to prove that every nonempty open set of the space $\mathcal{C}\big([0,1]^m\big)$ of all continuous self-maps on the unit cube $[0,1]^m$ in $\mathbb{R}^m$ contains a map that has not even discontinuous iterative roots of order $n \geq 2$. This, in particular, proves that continuous self-maps on $[0,1]^m$ with no continuous iterative roots are dense in $\mathcal{C}\big([0,1]^m\big)$. The talk is based on our recent works [1] and [2].

References

[1] Bhat, B. V. Rajarama, and Chaitanya Gopalakrishna. "Iterative square roots of functions." *Ergodic Theory Dynam. Systems* 43, no. 7 (2023): 2201-2227.
[2] Bhat, B. V. Rajarama, and Chaitanya Gopalakrishna. "The non-iterates are dense in the space of continuous self-maps." *Nonlinearity* 36, no. 6 (2023): 3419-3430.

Szymon Ignaciuk
University of Life Sciences in Lublin, Poland

Relationship between some angular inequalities and Kaplan classes for complex polynomials

(joint work with Maciej Parol)

The presented results concern a certain class of finite products of the form

$$\mathbb{D} \ni z \mapsto F_n(z; T_n; P_n) := \prod_{k=1}^{n} \left(1 - z^{-t_k}\right)^{p_k} \tag{1}$$

where $\mathbb{N} \ni n \mapsto T_n := (t_1, t_2, \ldots, t_n)$ is an increasing sequence of values from $[0; 2\pi)$ such that $t_1 := 0$ and $\mathbb{N} \ni n \mapsto P_n := (p_1, p_2, \ldots, p_n)$ is a sequence of real numbers of the same sign. The first results for polynomials with all zeros on unit circle $\mathbb{T}$ (when P_n is a sequence of natural numbers) were given by Jahangiri [4] in terms of a gap condition. Complete membership to Kaplan classes for the polynomials was presented in [1]. In the paper [2] the authors carried out complete membership to Kaplan classes of finite products of the form similar to (1), but with zeros symmetrically situated in $\mathbb{T}$. The presented results from the paper [3] generalize ones from [1] and [2]. An interpretation of the obtained gap condition in terms of mass and density is given in the case where all zeros of the studied functions are situated in $\mathbb{T}$.

An open problem is a more sophisticated physical interpretation that allows for the consideration of zeros outside the unit circle. A visualization of the obtained gap condition in terms of angular inequalities between vectors in $\mathbb{R}^2$ is given.

References

[1] Ignaciuk, Szymon, and Maciej Parol. "Zeros of complex polynomials and Kaplan classes." *Anal. Math.* 46, no. 4 (2020): 769-779.

[2] Ignaciuk, Szymon, and Maciej Parol. "Kaplan classes of a certain family of functions." *Ann. Univ. Mariae Curie-Skłodowska Sect. A* 74, no. 2 (2020): 31-40.

[3] Ignaciuk, Szymon, and Maciej Parol. "A gap condition for the zeros of a certain class of finite products." *Proc. Rom. Acad. Ser. A Math. Phys. Tech. Sci. Inf. Sci.* 22, no. 1 (2021): 19-25.

[4] Jahangiri, Massoud. "A gap condition for the zeroes of certain polynomials in Kaplan classes $K(\alpha, \beta)$." *Mathematika* 34, no. 1 (1987): 53-63.

Justyna Jarczyk
University of Zielona Góra, Poland

On a linear iterative equation of finite order and its solutions with a prescribed asymptotics

Fix a number $a \in (0, +\infty]$. We start with solutions $\varphi\colon (0,a) \to \mathbb{R}$ of the equation

$$\varphi(x) = \sum_{j=1}^{n} p_j(x)\varphi\big(f_j(x)\big), \tag{1}$$

where $p_1, \dots, p_n\colon (0,a) \to [0,+\infty)$ are given and $f_1, \dots, f_n\colon (0,a) \to (0,a)$ satisfy some natural assumptions and there exists a $\lambda \in \mathbb{N}_0$ such that

$$\sum_{j=1}^{n} p_j(x)\varphi\left(\frac{f_j(x)}{x}\right)^{\lambda} = 1$$

for all admissible x. Theorem 6.1.3 from the monograph [1] by Marek Kuczma, Bogdan Choczewski and Roman Ger shows that if φ satisfies (1) and the limit

$$\lim_{x\to 0} \frac{\varphi(x)}{x^{\lambda}} =: c$$

exists and is finite, then

$$\varphi(x) = cx^{\lambda}, \quad x \in (0,a).$$

We present a generalization of this result where it is enough to assume that λ is an arbitrary real number. It turns out that in fact something more can be proved. As consequences we obtain extensions of some results by Ernő Vincze, Marek Cezary Zdun, John A. Baker and, finally, Justyna and Witold Jarczyk.

References

[1] Kuczma, Marek, Bogdan Choczewski, and Roman Ger. *Iterative Functional Equations.* Vol. 32 of *Encyclopedia of Mathematics and its Applications.* New York: Cambridge Univ. Press, 1990.

Witold Jarczyk
University of Zielona Góra, Poland

Iterative roots of piecewise monotonic functions extended from the characteristic interval

(joint work with Veerapazham Murugan, Murugan Suresh Kumar and Justyna Jarczyk)

We consider the problem of finding iterative roots of order less than the number of forts of continuous piecewise monotonic functions with nonmonotonicity height greater than 1. We present sufficient conditions to extend iterative roots of piecewise monotonic functions from the characteristic interval which determines the behaviour of the function under iteration.

Divya Khurana
IIM Ranchi, India

Birkhoff-James orthogonality and its pointwise symmetry in some Banach spaces

Let X be a real Banach space. If $x, y \in X$, then we say that *x is Birkhoff-James orthogonal to y*, written as $x \perp_B y$, if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \in \mathbb{R}$. Generally, the Birkhoff-James orthogonality need not be symmetric, that is, $x \perp_B y$ does not necessarily imply that $y \perp_B x$. An element $x \in X$ is said to be *left-symmetric* (resp. *right-symmetric*), if the following condition holds:

$$x \perp_B y \ \text{ (resp. } y \perp_B x) \ \Rightarrow \ y \perp_B x \ \text{ (resp. } x \perp_B y) \ \text{ for all } \ y \in X.$$

In this talk, we will discuss symmetric, left-symmetric, and right-symmetric points in some Banach spaces.

Tibor Kiss
University of Debrecen, Hungary

Some solutions of the Járai–Maksa–Páles functional equation

The talk is about the solutions of the functional equation

$$F\Big(\frac{x+y}{2}\Big) + f_1(x) + f_2(y) = G\big(g_1(x) + g_2(y)\big), \qquad x, y \in I,$$

where I stands for an open interval and all functions are assumed to be unknown and continuously differentiable. The talk covers irregular solutions.

References

[1] Járai, Antal, Gyula Maksa, and Zsolt Páles. "On Cauchy-differences that are also quasisums." *Publ. Math. Debrecen* 65, no. 3-4 (2004): 381-398.
[2] Kiss, Tibor. "A Pexider equation containing the aritmetic mean." *Aequationes Math.*: accepted for publication.
[3] Kiss, Tibor. "Regular solutions of a functional equation derived from the invariance problem of Matkowski means." *Aequationes Math.* 96, no. 5 (2022): 1089-1124.

Milica Klaričić Bakula
University of Split, Croatia

Jensen-Steffensen Inequality: Accentuate the Negative

Let $f\colon I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$, be a convex function and let $\boldsymbol{x} = (x_1, \ldots, x_n) \in I^n$. If $\boldsymbol{p} = (p_1, \ldots, p_n)$ is a nonnegative real n-tuple such that $P_n = \sum_{i=1}^{n} p_i > 0$, then the well-known Jensen inequality

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) \tag{1}$$

holds. To Steffensen's credit, it is known that the assumption "$\boldsymbol{p}$ is a nonnegative real n-tuple" can be relaxed at the expense of further restrictions on the n-tuple $\boldsymbol{x}$. Namely, if $\boldsymbol{x} \in I^n$ is a monotonic (increasing or decreasing) n-tuple, then for any real n-tuple $\boldsymbol{p}$ such that

$$0 \leq P_j = \sum_{i=1}^{j} p_i \leq P_n,\ j = 1, \ldots, n, \qquad P_n > 0, \tag{2}$$

we get the condition

$$\overline{x} = \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i \in I,$$

and (1) still holds. The inequality (1) under the conditions (2) is known as the Jensen-Steffensen inequality for convex functions.

One can say that the Jensen-Steffensen inequality is "the ugly sister" of the Jensen inequality: not much admired and usually "not invited to the party". Our goal here is to show that "she" has many hidden beauties and that "she" can proudly walk hand in hand with her well-known sister.

Paweł A. Kluza
University of Life Sciences in Lublin, Poland

Inequalities for Jensen–Sharma–Mittal and Jeffreys–Sharma–Mittal type f–divergences

In this talk, we present new divergences called Jensen–Sharma–Mittal and Jeffreys–Sharma–Mittal in relation to convex functions. Some theorems, which give the lower and upper bounds for two new introduced divergences are provided. The obtained results imply some new inequalities corresponding to known divergences. Some examples, which show that it is generalization of Rényi, Tsallis and Kullback–Leibler types of divergences are provided in order to show a few applications of new divergences.

References

[1] Kluza, Paweł A. "Inequalities for Jensen-Sharma-Mittal and Jeffreys-Sharma-Mittal type f-divergences." *Entropy* 23, no. 12 (2021): Paper No. 1688.

[2] Kluza, Paweł A. "On Jensen-Rényi and Jeffreys-Rényi type f-divergences induced by convex functions." *Phys. A* 548 (2020): 122527.

[3] Kluza, Paweł A., and Marek Niezgoda. "On Csiszár and Tsallis type f-divergences induced by superquadratic and convex functions." *Math. Inequal. Appl.* 21, no. 2 (2018): 455-467.

[4] Kluza, Paweł, and Marek Niezgoda. "Generalizations of Crooks and Lin's results on Jeffreys-Csiszár and Jensen-Csiszár f-divergences." *Phys. A* 463 (2016): 383-393.

Tomasz Kochanek
University of Warsaw, Poland

Approximate multiplicativity and zero-product preserving in Banach algebras

Through last several decades, various aspects of Ulam stability have become fruitful areas of research in Banach algebra theory. Among them, there are three general subjects which appeared to be of special importance, and which are all underpinned by the two main themes mentioned in the title:

- approximately multiplicative maps and the theory of AMNM pairs,
- approximate representations and *-homomorphisms,
- almost zero-product-preserving maps.

The first subject was initiated by B.E. Johnson in *Memoirs AMS* (1972) by introducing cohomology theory of Banach algebras and notably revealing its connections with the notion of amenability, generalized from group theory to the Banach algebras setting. These new tools were then widely developed in his two seminal papers published in *J. London Math. Soc.* (1986–88), where he defined AMNM pairs as those pairs of Banach algebras for which an appropriate Ulam's stability effect holds true for approximate multiplicativity. The second subject has its roots in D. Kazhdan's stability theorem on approximate unitary representations (*Israel J. Math.* 1982), most noticeably used by I. Farah in his celebrated paper on automorphisms of the Calkin algebra published in *Ann. Math.* (2011). The last mentioned subject is a wide area of study concerning zero-product structures of Banach algebras, disjointness preserving operators and order zero maps, with some remarkable results obtained in commutative case by J. Araujo and J.J. Font (*J. London Math. Soc.* 2009–10), and for Fourier algebras, group algebras and C*-algebras by J. Alaminos, J. Extremera and A.R. Villena (*Israel J. Math.* 2010).

During the talk, we shall present an overview of the most important classical results in these topics, as well as a discussion of the newest results published in various papers during recent years by J. Alaminos, Y. Choi, J. Extremera, M. Ghandehari, M.L.C. Godoy, B. Horváth, N.J. Laustsen, H.L. Pham, A.R. Villena, and the author.

Dawid Komorek

AGH University of Science and Technology, Kraków, Poland

Continuous dependence of the weak limit of iterates of some random-valued vector functions

Given a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, a complete separable Banach space X with the σ-algebra $\mathcal{B}(X)$ of all its Borel subsets, an operator $\Lambda\colon \Omega \to L(X,X)$ and $\xi\colon \Omega \to X$ we consider the $\mathcal{B}(X) \otimes \mathcal{A}$-measurable function $f\colon X \times \Omega \to X$ given by $f(x,\omega) = \Lambda(\omega)x + \xi(\omega)$ and investigate the continuous dependence of the weak limit π^f of the sequence of iterates $\big(f^n(x,\cdot)\big)_{n\in\mathbb{N}}$ of f, defined by $f^0(x,\omega) = x$, $f^{n+1}(x,\omega) = f\big(f^n(x,\omega), \omega_{n+1}\big)$ for $x \in X$ and $\omega = (\omega_1, \omega_2, \ldots)$. Moreover for X taken as a Hilbert space we characterize π^f via the functional equation

$$\varphi^f(u) = \int_\Omega \varphi^f\big(\Lambda(\omega)u\big)\varphi^\xi(u)\mathbb{P}(d\omega)$$

with the aid of its characteristic function φ^f. We also indicate the continuous dependence of the solution of that equation.

References

[1] Komorek, Dawid. "Continuous dependence of the weak limit of iterates of some random-valued vector functions." *Aequationes Math.* 97, no. 4 (2023): 753-776.

Zbigniew Leśniak
Pedagogical University of Krakow, Poland

On foliations consisting of invariant lines of a Brouwer homeomorphism

We study properties of a Brouwer homeomorphism f for which there exists a foliation of the plane with leaves being invariant lines of f. For such a homeomorphism, we find the form of limit sets of strongly irregular points. Assuming the matching property for the leaves of the foliation contained in the set of strongly irregular points, we can construct continuous iterative roots of the Brouwer homeomorphism f.

References

[1] Gasull, Armengol, Jorge Groisman, and Francesc Mañosas. "Regularization of Brouwer translation theorem." *Qual. Theory Dyn. Syst.* 21, no. 4 (2022): Paper No. 135.
[2] Le Roux, Frédéric, Anthony Gilbert O'Farrell, Maria Roginskaya, and Ian Short. "Flowability of plane homeomorphisms." *Ann. Inst. Fourier (Grenoble)* 62, no. 2 (2012): 619-639.
[3] Leśniak, Zbigniew. "On families of invariant lines of a Brouwer homeomorphism." *J. Difference Equ. Appl.* 25, no. 9-10 (2019): 1363-1371.

Joanna Markowicz
Pedagogical University of Krakow, Poland

García-Falset coefficient and Opial property in direct sums of Banach spaces

We present some results on the the García-Falset coefficient and the Opial property in direct sums of Banach spaces. Constructing a general direct sum, we use a function lattice called a *substitution space* E. The *García-Falset coefficient* $R(X)$ *of a Banach space* X is a classical geometric parameter used in metric fixed point theory. In [1] J. García-Falset introduced it by

$$R(X) = \sup \left\{ \liminf_{n\to\infty} ||x + x_n|| \ : \ x, x_n \in B(X), \ x_n \overset{weak}{\longrightarrow} 0 \right\},$$

where $B(X)$ denotes the unit ball of X (see [1]). It characterizes the presence of a weak fixed point property for nonexpansive mappings in Banach spaces. We present an estimate for the García-Falset coefficient in the direct sum $Y = (\sum X_i)_E$ in terms of the supremum of $R(X_i)$ and the Riesz angle $\alpha(E)$ of the lattice E. Here, the Opial property is studied in the context of direct sums. We say that a Banach space X has the *non-strict Opial property* if

$$\liminf_{n\to\infty} ||x_n|| \leq \liminf_{n\to\infty} ||x_n - x||$$

for every weakly null sequence (x_n) in X and every $x \in X$. If

$$\liminf_{n\to\infty} ||x_n|| < \liminf_{n\to\infty} ||x_n - x||$$

for every weakly null sequence (x_n) in X and $x \in X \setminus \{0\}$, we say that X has the *Opial property*. We show that these properties are preserved under passing to direct sums. We present a new modulus corresponding to uniform Opial property and provide an estimate for the modulus of the direct sum in terms of the moduli of the spaces X_i and the modulus of monotonicity of E. These results contribute to a deeper understanding of direct sums of Banach spaces.

References

[1] García-Falset, Jesús. "Stability and fixed points for nonexpansive mappings." *Houston J. Math.* 20, no. 3 (1994): 495-506.

[2] Markowicz, Joanna, and Stanisław Prus. "James constant, García-Falset coefficient and uniform Opial property in direct sums of Banach spaces." *J. Nonlinear Convex Anal.* 17, no. 11 (2016): 2237-2253.

[3] Markowicz, Joanna, and Stanisław Prus. "Properties of modulus of monotonicity and Opial property in direct sums." *Ann. Univ. Mariae Curie-Skłodowska Sect. A* 71, no. 2 (2017): 69-77.

Kazuki Okamura
Shizuoka University, Japan

Properties of quasi-arithmetic means of random variables
(joint work with YUICHI AKAOKA and YOSHIKI OTOBE)

We will deal with quasi-arithmetic means of i.i.d. random variables. Here, we allow the target space to be the complex plane $\mathbb{C}$. This is useful for non-integrable heavy-tailed distributions supported on $\mathbb{R}$ such as Cauchy distributions. There are similarities and differences between the quasi-arithmetic mean of i.i.d. random variables and the arithmetic mean of i.i.d. random variables. We will focus on the limit theorem for variances and integrability. This talk is based on the papers [1] and [2].

References

[1] Akaoka, Yuichi, Kazuki Okamura, and Yoshiki Otobe. "Limit theorems for quasi-arithmetic means of random variables with applications to point estimations for the Cauchy distribution." *Braz. J. Probab. Stat.* 36, no. 2 (2022): 385-407.

[2] Akaoka, Yuichi, Kazuki Okamura, and Yoshiki Otobe. "Properties of complex-valued power means of random variables and their applications." *Acta Math. Hungar.*, to appear.

Andrzej Olbryś
University of Silesia, Katowice, Poland

On approximate convexity

Let D be a convex subset of a real linear space X. In this talk we examine the properties of functions $f\colon D \to \mathbb{R}$ satisfying the inequality

$$f\big(tx+(1-t)y\big) \le tf(x)+(1-t)f(y)+\phi\big(t(x-y)\big)-t\phi(x-y),$$

for all $x,y \in D,\ t\in[0,1]$, where $\phi\colon X \to \mathbb{R}$ is a given function.

Adam J. Ostaszewski
London School of Economics, United Kingdom

Homomorphisms from Functional Equations: The Goldie Equation Revisited

(joint work with Nicholas H. Bingham)

For X, Y real topological vector spaces, we characterize the continuous solutions of functional equation of the Levi-Civita-like equation

$$K\big(u + h(u)v\big) = K(u) + g(u)K(v) \qquad (u, v \in X), \qquad (GGE)$$

(cf. Goldie's equation in regular variation theory) as homomorphisms,

$$K\big(u \circ_\rho v\big) = K(u) \circ_\sigma K(v),$$

between 'Popa groups' generated by continuous linear maps $\rho \in X^*, \sigma \in Y^*$:

$$\mathbb{G}_\rho(X) := \big\{x \in X :\ h(x) = 1 + \rho(x) > 0\big\},$$
$$\mathbb{G}_\sigma(X) := \big\{y \in Y :\ g(y) = 1 + \sigma(y) > 0\big\}.$$

The binary operation $u \circ_\rho v = u + \big(1 + \rho(u)\big)v$ creates group structure. The key to the characterization are two collaborative dichotomies:

(i) homeomorphy implies one or other of the inclusions $K\big(\mathcal{N}(\rho)\big) \subseteq \mathcal{N}(\sigma)$ or $K\big(\mathcal{N}(\rho)\big) \subseteq \mathrm{Lin}K(w)$ for some w; and

(ii) the intersection of two hyperplanes is of codimension 1 or 2.

Some of the results require the inner auxiliary function h to be Gateaux differentiable at the origin.

Zdzisław Otachel
University of Life Sciences in Lublin, Poland

Jensen-Jessen inequality for sublinear functionals

The notion of convex function ϕ is inextricably linked with the inequality

$$\phi\left(\sum_{k=1}^{n} w_k x_k\right) \leq \sum_{k=1}^{n} w_k \phi(x_k),$$

where ϕ is defined on an interval containing real numbers x_k, while w_k, $(k = 1, \dots, n)$ are positive weights with the sum equals 1, in the sense that the inequality can be treated as a definition of a convex function itself. Counterparts of the inequality for integrals also hold true. The introduced facts were established by Jensen [2].

Jessen [3] unified these results replacing sums and integrals by an abstract linear and positive functional A acting on a linear space of real functions g including constant functions, such that $A(J) = 1$, where J is the function constantly equals 1. More precisely, he showed that

$$\phi\big(A(g)\big) \leq A\big(\phi(g)\big).$$

Functionals of that type are called *linear means.*

The Jensen-Jessen inequalities remain true for the wider class of functionals than linear means, namely, sublinear isotonic functionals preserving constants. This result comes from Dragomir, Pearce and Pečarić [1].

We show that Jensen-Jessen inequalities are met with weaker assumptions about functionals than those made in [1]. This generalizes Jessen's inequality and complements the result by Dragomir, Pearce and Pečarić.

References

[1] Dragomir, Silvestru Sever, Charles Edward Miller Pearce, and Josip Pečarić. "On Jessen's and related inequalities for isotonic sublinear functionals." *Acta Sci. Math. (Szeged)* 61, no. 1-4 (1995): 373-382.
[2] Jensen, Johan Ludwig William Valdemar. "Sur les fonctions convexes et les inéqualités entre les valeurs moyennes." *Acta Math.* 30, no. 1 (1906): 175-193.
[3] Jessen, Børge. "Bemaerkinger om konvekse Funktioner og Uligheder imellem Middelvaerdier I." *Mar. Tidsskrift, B* (1931): 17-28.

Paweł Pasteczka
Pedagogical University of Krakow, Poland

Invariance property for extended means

We study the properties of the mean-type mappings $\mathbf{M}\colon I^p \to I^p$ of the form

$$\mathbf{M}(x_1,\dots,x_p) := \big(M_1(x_{\alpha_{1,1}},\dots,x_{\alpha_{1,d_1}}),\dots,M_p(x_{\alpha_{p,1}},\dots,x_{\alpha_{p,d_p}})\big),$$

where p and d_i-s are positive integers, each M_i is a d_i-variable mean on an interval $I \subseteq \mathbb{R}$, and $\alpha_{i,j}$-s are elements from $\{1,\dots,p\}$.

We show that under some natural assumption on M_i-s, the problem of existing the unique $\mathbf{M}$-invariant mean can be reduced to the ergodicity of the directed graph with vertices $\{1,\dots,p\}$ and edges $\big\{(\alpha_{i,j},i)\ :\ i,j \text{ admissible}\big\}$.

This talk is based on the recent article [1].

References

[1] Pasteczka, Paweł. "Invariance property for extended means." *Results Math.* 78, no. 4 (2023): Paper No. 146.

Kallol Paul
Jadavpur University, Kolkata, India

Geometry of the space of linear operators through the lens of Birkhoff-James orthogonality

The Birkhoff-James orthogonality generalizes the concept of usual orthogonality in an Euclidean space. In this talk we discuss the Birkhoff-James orthogonality of bounded linear operators defined between Banach spaces and investigate the role of Birkhoff-James orthogonality in the study of geometric properties of the space of bounded linear operators. We also plan to talk about some open problems in this area.

Teresa Rajba
University of Bielsko-Biala, Poland

On the Raşa, Hermite-Hadamard and Jensen inequalities for Popoviciu's box-(m,n)-convex functions

(joint work with Andrzej Komisarski)

This talk is based on the article [3]. We give the integral representation of Popoviciu's [4] box-(m,n)-convex functions. Based on this integral representation, we obtain a characterization of box-(m,n)-convex functions, which we then use in the proofs of the Raşa, Hermite-Hadamard type and Jensen inequalities for box-(m,n)-convex functions.

We extend the results obtained by S. G. Gal and C. P. Niculescu [1], and by B. Gavrea and I. Gavrea [2].

References

[1] Gal, Sorin Gheorghe, and Constantin P. Niculescu. "A new look at Popoviciu's concept of convexity for functions of two variables." *J. Math. Anal. Appl.* 479, no. 1 (2019): 903-925.

[2] Gavrea, Bogdan, and Ioan Gavrea. "An inequality involving Bernstein polynomials and box-convex functions." *Mediterr. J. Math.* 19, no. 1 (2022): Paper No. 18.

[3] Komisarski, Andrzej, and Teresa Rajba. "On the integral representation and the Raşa Jensen and Hermite-Hadamard inequalities for box-convex functions." *Results Math.* 78, no. 3 (2023): Paper No. 89.

[4] Popoviciu, Tibère. "Sur quelques propriétés des fonctions d'une ou de deux variables réelles". *Mathematica* 8 (1934): 1-85.

Ioan Raşa
Technical University of Cluj-Napoca, Romania

Kantorovich type operators and convexity

We introduce a sequence of Kantorovich type operators which preserve the affine functions. They are defined as

$$\mathcal{K}_n f(x) = f(0)p_{n,0}(x) + f(1)p_{n,n}(x) + \frac{1}{2a_n}\sum_{k=1}^{n-1} p_{n,k}(x) \int_{\frac{k}{n}-a_n}^{\frac{k}{n}+a_n} f(t)dt,$$

where $f \in C[0,1]$, $p_{n,k}(x) = \binom{n}{k} x^k(1-x)^{n-k}$, $x \in [0,1]$ and $0 < a_n \le \dfrac{1}{n}$.
In the limiting case $a_n \to 0$ we get the classical Bernstein operators.

Some approximation properties are investigated. Moreover, we present inequalities involving the new operators and convex functions.

Jens Schwaiger
University of Graz, Austria

The Sincov equation, history, applications, and new developments

After a short excursion to the history of the Sincov equation as described by Detlef Gronau in [*A remark on Sincov's functional equation*, Notices of the South African Mathematical Society, **31**(1), (2000), 1–8], some applications, old and recent ones, are presented. Finally a result of Prasanna K. Sahoo on the Sincov like functional equation

$$f_1(x,y)+f_2(x,z)+f_3(z,x)=f(x+y+z).$$

is generalized by considering

$$\sum_{i=1}^{n} f_i(x_i,x_{i+1}) = f\left(\sum_{i=1}^{n} x_i\right) \quad \text{with} \quad x_{n+1}=x_1,$$

where f_i, f are functions defined on an Abelian group G and taking values in an Abelian group H, and by determining its solutions.

Wutiphol Sintunavarat
Thammasat University, Pathum Thani, Thailand

The modification of fuzzy numbers with its impact on the stability of fuzzy number-valued functional equations

In this talk, we will provide remarks on the definition of fuzzy numbers and discuss various stability results of fuzzy number-valued functional equations. Building upon these observations, we will introduce a slight modification of fuzzy numbers and explore its implications on numerous stability results of fuzzy number-valued functional equations.

Aleksandra Świątczak

Lodz University of Technology, Poland

Functional equations characterizing differential operators

(joint work with Włodzimierz Fechner and Eszter Gselmann)

If $k \geq 2$ is a positive integer and $\Omega \subseteq \mathbb{R}^N$ is a domain, then by the well-known properties of the Laplacian and the gradient, we have

$$\Delta(f \cdot g) = g\Delta f + f\Delta g + 2\langle \nabla f, \nabla g \rangle$$

for all $f, g \in \mathcal{C}^k(\Omega, \mathbb{R})$. Due to the results of König–Milman [1], the converse is also true under some assumptions. Thus the main aim of this talk is to study the equation

$$T\big(f \cdot g\big) = fT(g) + T(f)g + 2B\big(A(f), A(g)\big) \qquad (f, g \in P)\,,$$

where Q and R are commutative rings and P is a subring of Q, further $T\colon P \to Q$ and $A\colon P \to R$ are additive mappings, while $B\colon R \times R \to Q$ is a symmetric and bi-additive mapping.

References

[1] König, Herman, and Vitali Milman. *Operator Relations Characterizing Derivatives.* Berlin: Springer, 2018.

László Székelyhidi

University of Debrecen, Hungary

New Results on Spectral Synthesis

In this talk we present some new results related to spectral synthesis on not necessarily discrete groups. We exhibit a new method of proving spectral synthesis on varieties which depends on the Fourier algebra of the underlying group. We show that derivations on the Fourier algebra play a deciding role when investigating spectral synthesis. We show how known results can be proved using this method, and also how new results can be obtained.

Patricia Szokol
University of Debrecen, Hungary

G-majorization and Schur-type convexity

(joint work with Pál Burai)

In this presentation we recall the standard majorization of vectors (introduced by Hardy, Littlewood and Pólya), that is closely related to Schur-convex functions. We present G-majorization, i.e. majorization induced by a linear group operating on $\mathbb{R}^n$. As a particular case (when G is the group of cyclic permutations), we obtain the notion of cyclic majorization.

As a consequence, we get the definition of cyclically convex functions that can be considered as a generalization of Schur-convex functions. Finally, we present characterization theorems of such kind of functions.

Bettina Wilkens
University of Namibia, Namibia

An alternative additive-quadratic equation

(joint work with Gian-Luigi Forti)

Parts of the result have been presented at ISFE in Innsbruck last year. Let G be a group and H be an abelian group. A map $f\colon G \to H$ is said to satisfy the *alternative additive and quadratic equation* if

$$f(xy) = f(x) + f(y) \text{ or } f(xy) + f\left(xy^{-1}\right) = 2f(x) + 2f(y) \qquad (*)$$

holds for $x, y \in G$. In the talk we present a complete description of the case where G, too, is abelian.

Theorem. *Let G and H be abelian groups and let $f\colon G \to H$ be a map satisfying equation $(*)$. Assume that $H = \langle f(G)\rangle$. The map f satisfies the additive or the quadratic functional equation unless $H \cong \langle\alpha\rangle + K$ with $2K = 0$, $\langle\alpha\rangle \cong \mathbb{Z}/8\mathbb{Z}$ and there is an element x of G such that*

$$f(kx) = \begin{cases} 0 & \textit{if } k \equiv 0 \pmod 4, \\ \alpha & \textit{if } k \equiv 1 \pmod 4, \\ 4\alpha & \textit{if } k \equiv 2 \pmod 4, \\ 5\alpha & \textit{if } k \equiv 3 \pmod 4. \end{cases}$$

We will provide details of the proof as well as more detailed description of the one "truly alternative" case. Possible applications to some nonabelian cases are briefly discussed.

References

[1] de Place Friis, Peter, and Henrik Stetkær. "On the quadratic functional equation on groups." *Publ. Math. Debrecen* 69, no. 1-2 (2006): 65-93.

Sebastian Wójcik
University of Rzeszów, Poland

Zero utility principle under uncertainty

A process of insurance contract pricing consists on assigning to any risk, represented by a non-negative essentially bounded random variable on a given probability space, a non-negative real number, being a premium for the risk. In a literature one can find various methods of insurance contracts pricing. In this paper we consider the method, known as the zero utility principle, introduced by H. Bühlmann [1]. It presents the problem from the point of view of an insurance company, assuming that the premium for a given risk is determined in such a way that the company is indifferent between entering into contract and rejecting it.

We study the zero utility principle in the cumulative prospect theory (cf. [2]) under uncertainty. In this setting, the risks are represented by measurable function defined on a given measurable space $(S, \mathcal{F})$. The principle for a risk X is defined as a real number $H_{(u,\mu,\nu)}(X)$ satisfying the equation

$$E_{\mu\nu}\big[u\big(H_{(u,\mu,\nu)}(X) - X\big)\big] = 0,$$

where $E_{\mu\nu}$ is the Choquet integral with respect to $\mu, \nu \colon \mathcal{F} \to [0,1]$.

We establish a necessary and sufficient condition for existence of the principle. Furthermore, we characterize its important properties: the comparability, equality, positive homogeneity and comonotonic additivity.

References

[1] Bühlmann, Hans. *Mathematical Models in Risk Theory.* Vol. 172 of *A series of Comprehensive Studies in Mathematics.* Berlin: Springer-Verlag, 1970.

[2] Tversky, Amos, and Daniel Kahneman. "Advances in prospect theory: Cumulative representation of uncertainty." *J. Risk Uncertain.* 5, no. 4 (1992): 297-323.

Marek Cezary Zdun
Pedagogical University of Krakow, Poland

On Koopmans recursion

Let I be an interval and X be a metric space. Assume that $U\colon X^\infty \to I$ is a continuous surjection satisfying the *Koopmans recursion*

$$U(x_0, x_1, x_2, \ldots) = f_{x_0}\big(U(x_1, x_2, \ldots)\big), \quad (x_0, x_1, \ldots) \in X^\infty,$$

where $f_x\colon I \to I$ are strictly increasing functions such that the mapping $(x,t) \to f_x(t),\ x \in X,\ t \in I$ is continuous. The mapping U is said to be the *utility function*. We show that every function f_x, $x \in X$ has a unique attractive fixed point and $U(x_0, x_1, \ldots) = \lim\limits_{n\to\infty} \big(f_{x_0} \circ f_{x_1} \circ \ldots \circ f_{x_n}\big)(t)$ for every $t \in I$. Moreover, define on X^∞ the order relation

$$(x_0, x_1, \ldots) \succeq (y_0, y_1, \ldots) \quad \Leftrightarrow \quad U(x_0, x_1, \ldots) \geq U(y_0, y_1, \ldots).$$

This order is called the *preference relation represented by* U. The existence of homeomorphic solutions of the system of simultaneous functional equations

$$\varphi(f_x(t)) = \alpha_x \varphi(t) + \beta_x, \quad x \in X,\ t \in I \tag{1}$$

is equivalent to a special property of the preference relation $\succeq$ represented by U. We determine the form of all f_x which ensures the existence of homeomorphic solution of (1). We show that the homeomorphic solution φ of (1) and coefficients $\alpha_x,\ \beta_x$ determine the form of the utility function U. The problem of the uniqueness of homeomorphic solutions of system (1) is considered. Some applications in the preference theory in economics will be presented.